The Valuation and Hedging Strategy of High Yield Notes

Szu-Lang Liao *
Department of Money and Banking
National Chengchi University

Chou-Wen Wang
Department of Finance
National Kaohsiung First University of Science and Technology

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* Correspondence: Szu-Lang Liao, Department of Money and Banking, National Chengchi University, Taipei 116, Taiwan. Tel: (02) 2939-3091 ext. 81251; Fax: (02) 2939-8004; E-mail: liaosl@nccu.edu.tw.
ABSTRACT

We derive a closed-form pricing formula for high yield notes that have been planned to be issued by Taiwanese local security firms in 2003. These financial derivatives are pure discount bonds with an embedded short position in equity put options. They are useful hedging instruments in financial markets with a legal restriction on issuing put options on equities or under bear market conditions. We show that high yield notes can be synthesized by buying pure discount bonds and selling exchange-linked put options, and investigate their specific properties and the corresponding hedging strategy for issuing high yield notes. Finally, we derive the closed-form pricing formula and hedging strategy for high yield notes under the Gaussian HJM framework of stochastic interest rates.
1. INTRODUCTION

High yield notes are one form of equity-linked structured notes which are fixed income securities whose interest coupons and/or principal is linked to the movements in equity prices. The underlying equity prices may be the values of individual equity securities or equity market indexes. As with other forms of structured notes, the equity-linked structured notes can typically be decomposed into two major instruments: a fixed income security and an equity derivative embedded in the notes. Various forms of equity-linked notes include convertible bonds, bond-with-equity warrants, convertible preferred stocks, Liquid Yield Option Notes (LYONs), exchangeable bonds, going-public bonds, high yield notes and principal-protected notes.

Some equity-linked notes, such as convertible bonds and bond-with-equity warrants, have a long history, while other types of equity-linked notes are the products of recent financial innovations. One of these new instruments is high yield notes, which is a reverse convertible bond paying a higher than market coupon or higher yield in return for the investor in the note granting a put or call option on a nominated stock. More specifically, the payoff at maturity of a high yield note is a constant amount of face value denominated in some currency (foreign or domestic) if the share price is higher than some pre-specified strike price; otherwise the investor receives the value of the predetermined number of shares. A high yield note combing a zero coupon bond with an embedded call option can be defined as above with the state conditions reversed.

High yield notes were developed in 1993 by Bankers Trust. The market spread in the late 1990s to Europe and Asia due to the low interest yield and high equity volatility. In 2003, Taiwan’s government has planned to legally allow local security dealers to issue high yield notes. In fact, high yield notes denominated in foreign currency have been prevalent in Taiwan’s financial market for many years. In view of the important role played by the high yield notes in world financial markets as well as in Taiwan’s market, in this paper we intend to provide closed-form solutions and hedging strategies for issuing high yield notes.

An outline of the article is as follows. Section 2 presents the notation and a brief definition of the payoffs of high yield notes. A literature review is provided in Section 3. We derive the closed-form solutions of exchange-linked stock options and high
yield notes in Section 4. In Section 5, we develop a hedging strategy for issuing high yield notes. In Section 6, we present the numerical results of high yield notes issued by ABN AMRO bank on June 18, 2001. We also show some properties of high yield notes and the corresponding hedging portfolio using domestic underlying stock and foreign pure discount bonds to hedge the risk exposure of issuing high yield notes. In Section 7, we derive the pricing formula of high yield notes under a stochastic interest rate environment. Section 8 develops a hedging strategy for issuing high yield notes under stochastic interest rates. We present the numerical pricing results under stochastic interest rates in Section 9. A brief summary and conclusion is offered in Section 10.

2. NOTATION AND PAYOFF DEFINITION

For the ease of exposition, let us first introduce some notations to make the payoffs of high yield notes clear. Suppose the time interval for the economy is $[0, T^*]$. At time $t \in [0, T]$, we assume that $S_d(t)$ is the domestic equity price and $Q_f(t)$ is the exchange rate, which is the domestic currency price per unit of foreign currency. Let $T_D \in [t, T]$ be the decision date of the equity-linked note with maturity date $T$. The payoff denominated in foreign currency at time $T$ of high yield note $C(T)$ is as follows:

$$C(T) = \begin{cases} 
M_f Q_f(T), & \text{if } S_d(T_D) \geq K; \\
\frac{M_f Q_f(T)}{K} S_d(T), & \text{if } S_d(T_D) < K,
\end{cases}$$

(1)

where $M_f$ is the foreign-currency denominated principal and $K$ is the strike price in domestic currency. Let $ELC(T)$ and $ELP(T)$ be the payoffs on expiry date $T$ of the exchange-linked call and put options as follows:

$$ELC(T) = Q_f(T) \text{Max}[S_d(T) - K, 0], \quad ELP(T) = Q_f(T) \text{Max}[K - S_d(T), 0].$$

(2)

Consider the case $T_D = T$ of equity-linked notes, which is often adopted in practice.

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1 If the payoff is denominated in domestic currency, the exchange rate $Q_f(t) \equiv 1$ and its volatility is zero.
We can replicate the terminal payoff of a high yield note as specified in (1) by buying $M_f$ units of foreign pure discount bonds, denoted as $B_f(t, T)$, which pays one unit of foreign currency at date $T$, and short selling $M_f/K$ units of exchange-linked put options. Therefore, the price at time $t$ of a high yield note under the special case of $T_D = T$ can be rewritten as follows:

$$C(t) = M_fB_f^*(t, T) - M_fKELP(t),$$  \hspace{1cm} (3)$$

where $B_f^*(t, T) = Q_f(t)B_f(t, T)$, i.e. the domestic-currency price of a foreign pure discount bond. Due to the fact that the solution of $B_f(t, T)$ is available, the closed-form pricing formula of a high yield note is determined by the closed-form valuation of an exchange-linked put option.

3. LITERATURE REVIEW

For related currency derivatives pricing, Biger and Hull (1983) and Garman and Kohlhagen (1983) have shown that if the domestic and foreign interest rates are constant, then the foreign currency option can be valued by means of a suitable modification of the Black-Scholes formula. Reiner (1992) explains how to adapt the Black-Scholes formula and its variants for four scenarios. The first scenario is a foreign equity call option struck in foreign currency, with terminal payoff of

$$C^1_T = Q_f(T)\text{Max}[S_f(T) - K_f, 0],$$

where $S_f(T)$ is the foreign equity price in its own currency at time $T$; $K_f$ is a foreign currency amount. The second case is a foreign equity call option struck in domestic currency. The payoff on the expiry date is

$$C^2_T = \text{Max}[Q_f(T)S_f(T) - K, 0],$$

where $K$ is now a domestic currency amount. The third payoff is a foreign equity call option under a fixed exchange rate, also known as a Quanto, and is given by
where $\overline{Q}$ is the pre-specified exchange rate. The last scenario is an equity-linked foreign exchange call option. The terminal payoff is

$$C_T^3 = \overline{Q}\text{Max}[S_f(T) - K, 0],$$

In this article, we provide a closed-form valuation for another scenario: exchange-linked equity options as specified in (2). After the closed-form solution of the exchange-linked put option has been found, we then use it to derive the prices of high yield notes.

The research on currency option pricing has evolved from a constant interest rate assumption to a stochastic interest rates environment, with the latter represented by Grabbe (1983), Amin and Jarrow (1991), and Tucker and Wei (1998). Numerous empirical studies such as Hillard et al. (1991) find that a stochastic interest rates model is superior to a constant interest rate model. Amin and Bodurtha (1995) find that interest rate risk parameters can significantly affect the values of long-term currency put warrants. Therefore, we also provide the closed-form solution and hedging strategies for high yield notes under the Gaussian Heath, Jarrow, and Morton (1992, henceforth HJM) framework.

4. PRICING EXCHANGE-LINKED STOCK OPTIONS AND HIGH YIELD NOTES

In this section, we introduce the trading economy and derive the closed-form solutions of exchange-linked call and put options on equities and their put-call parity. Then, we derive the closed-form solution of high yield notes.

We assume that the dynamics of domestic underlying asset price $S_d(t)$ are described by the following linear stochastic differential equation:

$$\frac{dS_d(t)}{S_d(t)} = (u - \delta)dt + \sigma_S \cdot dW_t,$$

where $u$ is the expected return and $\sigma_S$ is an $h$-dimensional vector of volatilities. $\delta$ rep-
represents the continuous dividend payout rate. $W_t$ stands for an $h$-dimensional standard Brownian motion defined on a filtered probability space $(\Omega, F, P, (F_t)_{t=0}^{T^*})$. The money market account, $B_d(t)$ and $B_f(t)$ are denominated in the units of domestic and foreign currency, respectively. They correspond to the wealth accumulated by an initial one-dollar investment at domestic spot interest rates $r_d$ or foreign spot interest rates $r_f$ in each subsequent period. Therefore,

$$dB_i(t) = r_iB_i(t)dt, \quad i = d, f.$$ 

The exchange rate process, $Q_f(t)$, which is used to convert foreign payouts into domestic currency, is assumed to follow stochastic differential equation

$$\frac{dQ_f(t)}{Q_f(t)} = u_Q dt + \sigma_Q \cdot dW_t,$$

where $\sigma_Q$ is an $h$-dimensional vector of volatilities. Assume that there exists a unique domestic spot martingale measure $P_d^*$ on $(\Omega, F)$, which is given by the Radon-Nikodym derivative

$$\frac{dP^{d^*}}{dP} = \exp \left( \eta \cdot W_{T^*} - \frac{1}{2} |\eta|^2 T^* \right),$$

where $\eta \in R^h$ is the vector of market prices of risks corresponding to the sources of randomness in the economy; $| \cdot |$ denotes the Euclidean norm in $R^h$. By definition, $\eta$ satisfies

$$u - \delta - r_d + \sigma_S \cdot \eta = 0, \quad (4)$$

$$u_Q + r_f - r_d + \sigma_Q \cdot \eta = 0. \quad (5)$$

Under the domestic spot martingale measure $P^{d^*}$, the dynamics of domestic underlying asset price $S_d(t)$ and the exchange rate $Q_f(t)$ become

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2 The uniqueness of a risk-neutral probability measure holds if the number of non-redundant traded assets equals $h + 1$. But the uniqueness of solution of (4) and (5) need not hold, in general.
\[
\frac{dS_d(t)}{S_d(t)} = (r_d - \delta)dt + \sigma_S \cdot dW_t^d, \\
\frac{dQ_f(t)}{Q_f(t)} = (r_d - r_f)dt + \sigma_Q \cdot dW_t^d,
\]

where the process \(W_t^d\) is defined as
\[
dW_t^d = dW_t - \eta dt.
\]

By Girsanov’s theorem, \(W_t^d\) is an \(h\)-dimensional standard Brownian motion under \(P^d\). In view of (2), the valuation of the exchange-linked put option is given by
\[
\text{ELP}(t) = \exp[-r_d(T - t)]E_{P^d}[\text{ELP}(T)|F_t] \\
= \exp[-r_d(T - t)]\{E_{P^d}[KQ_f(T)I[K \geq S_d(T)]|F_t] - \\
E_{P^*}[Q_f(T)S_d(T)I[K \geq S_d(T)]|F_t]\}, \quad (6)
\]

where \(I(\cdot)\) are the indicator functions. We provide the closed-form solutions of exchange-linked options and the put-call parity in Theorem 1.

**Theorem 1**  
The closed-form solution of the exchange-linked put option is as follows:

\[
\text{ELP}(t) = Q_f(t)\exp[-r_f(T - t)]\{-S_d(t)\exp[(\rho SQ\hat{\sigma}_S\hat{\sigma}_Q + r_d - \delta)(T - t)]} \\
N(-d_1) + KN(-d_2)\}. \quad (7)
\]

The put-call parity of exchange-linked stock options is
\[
Q_f(t)S_d(t)\exp[(\rho SQ\hat{\sigma}_S\hat{\sigma}_Q + 2r_d - r_f - \delta)(T - t)] + \text{ELP}(t) - \text{ELC}(t) \\
= KQ_f(t)\exp[-r_f(T - t)]. \quad (8)
\]

Accordingly, the value of \(\text{ELC}(t)\) is given by
\[
\text{ELC}(t) = Q_f(t)\exp[-r_f(T - t)]\{S_d(t)\exp[(\rho SQ\hat{\sigma}_S\hat{\sigma}_Q + r_d - \delta)(T - t)] \\
N(d_1) - KN(d_2)\}, \quad (9)
\]

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where \( N(\cdot) \) is the cumulative probability of a standard normal distribution and

\[
d_{1,2} = \ln \left[ \frac{S_d(t)}{K} \right] + \left( r_d - \delta + \rho_{SQ} \hat{\sigma}_S \hat{\sigma}_Q \pm \frac{1}{2} \hat{\sigma}_S^2 \right) (T_D - t) \frac{\hat{\sigma}_S \sqrt{I_D - t}}{\hat{\sigma}_D} ;
\]

\( \rho_{SQ} \) is the correlation coefficient between the domestic stock price and exchange rate and is defined as follows:

\[
\rho_{SQ} = \frac{\sigma_S \cdot \sigma_Q}{\hat{\sigma}_S \hat{\sigma}_Q}, \quad \hat{\sigma}_S = |\sigma_S|, \quad \hat{\sigma}_Q = |\sigma_Q|.
\]

The proof of Theorem 1 is shown in Appendix.

Now we derive the closed-form solution of the high yield note. In view of (1), the value of the high yield note is as follows:

\[
C(t) = \exp[-r_d(T - t)]E_{\mathcal{F}_t}[C(T) | F_t]
\]

\[
= \exp[-r_d(T - t)]E_{\mathcal{F}_t}\left[ M_f Q_f(T) I[S_d(T_D) \geq K] + \frac{M_f Q_f(T) S_d(T)}{K} I[S_d(T_D) < K] | F_t \right] .
\]

(10)

We provide the closed-form solution of the high yield note in Theorem 2.

**Theorem 2**  The closed-form pricing formula of the high yield note is as follows:

\[
C(t) = M_f Q_f(t) \exp[-r_f(T - t)] \left\{ N(d_2) + \frac{S_d(t)}{K} \exp[(\rho_{SQ} \hat{\sigma}_S \hat{\sigma}_Q + \delta) (T - t)] N(-d_1) \right\},
\]

(11)

where \( N(\cdot) \) and \( d_{1,2} \) are defined as in Theorem 1.

We prove Theorem 2 in Appendix.

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3 In this paper, we compute the values of contingent claims in units of domestic currency.
Consider the special case \( T_D = T \) of high yield notes. We can synthesize it by buying \( M_f \) units of foreign pure discount bonds and selling \( M_f/K \) units of exchange-linked put options. Since the domestic-currency value of the foreign pure discount bond is

\[ B_f^* (t, T) = \exp[-r_d(T - t)] E_{\delta t^*} [Q_f(T)|F_t] = Q_f(t) \exp[-r_f(T - t)], \]

therefore, using the results from Theorems 1 and 2, we can check that equation (3) is satisfied under the special case of \( T_D = T \).

5. HEDGING STRATEGY FOR HIGH YIELD NOTES

In this section, we show that the appropriate hedging strategies for the issuers after they have issued high yield notes. To hedge the high yield notes, we construct a hedged portfolio by short selling a high yield note and buying \( m \) units of the domestic underlying assets \( S_d(t) \) and \( n \) units of the foreign pure discount bonds \( B_f(t, T) \). The value of this portfolio \( W(t) \) is

\[ W(t) = -C(t) + mS_d(t) + nQ_f(t)B_f(t, T). \]

Since \( C(t) \) is a function of \( S_d(t), Q_f(t) \), and time \( t \), by Ito’s lemma we obtain

\[ dW(t) = -dC(t) + mdS_d(t) + n(Q_f(t)dB_f(t, T) + B_f(t, T)dQ_f(t)) \]

\[ = W_G dt + (-C_S S_d(t)\sigma_S - C_Q Q_f(t)\sigma_Q + mS_d(t)\sigma_S \]

\[ + nQ_f(t)B_f(t, T)\sigma_Q) \cdot dW_t^\delta, \]

where

\[ C_i = \frac{\partial C(t)}{\partial i}, \quad C_t = \frac{\partial C(t)}{\partial t}, \quad C_{ij} = \frac{\partial^2 C(t)}{\partial i \partial j}, \quad i, j = S \text{ or } Q. \]
\[ W_G = -C_G + mS_d(t)(r_d - \delta) + nQ_f(t)B_f(t,T)r_d, \]

\[ C_G = S_d(t)C_S(r_d - \delta) + C_t + Q_f(t)C_Q(r_d - r_f) \]
\[ + \frac{1}{2} \left[ C_Q Q_f(t)^2 \sigma_Q^2 + C_{SS} S_d(t)^2 \sigma_S^2 \right] + C_{SQ} Q_f(t) S_d(t) \rho S \hat{\sigma} S \hat{\sigma}_Q. \]

When \( m = C_S \) and \( n = C_Q / B_f(t, T) = C_Q \exp[r_f(T - t)] \), the portfolio is riskless. For hedging purpose, the issuers of high yield notes, such as investment banks, should buy \( C_S \) units of the domestic underlying assets and \( C_Q \exp[r_f(T - t)] \) units of the foreign pure discount bonds. From Theorem 1 and the chain rules

\[ \frac{\partial N(x)}{\partial S_d(t)} = \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{x^2}{2} \right) \times \frac{\partial x}{\partial S_d(t)}, \quad \frac{\partial d_{1,2}}{\partial S_d(t)} = \frac{1}{S_d(t) \hat{\sigma}_S \sqrt{T_D - t}}, \]

we have

\[ C_S = MQ_f(t) \]
\[ \begin{cases} 
\frac{1}{S_d(t) \hat{\sigma}_S \sqrt{2\pi(T_D - t)}} \exp\left[ -r_f(T - t) - \frac{d_2^2}{2} \right] \\
- \frac{1}{K \hat{\sigma}_S \sqrt{2\pi(T_D - t)}} \exp\left[ \eta(T - t) - \frac{d_1^2}{2} \right] + \frac{N(-d_1)}{K} \exp[\eta(T - t)] 
\end{cases}, \]

\[ C_Q = \frac{C(t)}{Q_f(t)}, \]

where \( \eta = \rho SQ \hat{\sigma} S \hat{\sigma}_Q + r_d - r_f - \delta. \)

In sum, for the issuers of high yield notes, on each monitoring date, they should buy \( C_S \) units of the domestic underlying assets and \( C_Q \exp[r_f(T - t)] \) units of the foreign pure discount bonds, as defined in (13) and (14), to construct a hedging portfolio to avoid the risk induced by issuing one unit of the high yield note.
6. NUMERICAL ANALYSES OF HIGH YIELD NOTES

To investigate the properties of high yield notes, we take a 3-month high yield note issued by ABN AMRO bank on June 18, 2001 as an example. The underlying stock is UMC (2303 in Taiwan Stock Exchange). The par value is USD (U.S. dollar) 10,000 per note and the maturity date is September 20, 2001. The strike price is 85% of the reference spot price TWD (Taiwan dollar) 50. The exchange rate on June 18, 2001 is about 34.46. If the final spot price is equal to or greater than the strike price, cash settlement of par value in units of USD will be paid on the maturity date. Otherwise, the holders of the high yield notes will receive a corresponding equity certificate of underlying stock shares, which is determined by the prevailing exchange rate and the spot price of the determination date, September 18, 2001. We summarize the numerical results in Table 1.

From Table 1, we see that the value of the high yield note is an increasing function of the domestic risk-free interest rate, the instantaneous correlation coefficient of stock and exchange rate $\rho_{SQ}$, and is a decreasing function of the volatility of stock returns and the foreign risk-free interest rate. However, the sign of instantaneous correlation coefficient determines the impact of the volatility of the exchange rate on the value of the high yield note. When $\rho_{SQ} > 0$, the value of the high yield note is increasing with the volatility of the exchange rate. If $\rho_{SQ} < 0$, the value of the high yield note decreases with the increase of the volatility of the exchange rate.

From the point of view of the issuers of high yield notes, Table 1 has shown the arbitrage-free values of high yield notes as a benchmark for issuing price. The next step for issuers is to hedge the risk exposure induced by the issue of high yield notes. Presently, we assume that the domestic and foreign risk-free rates are 3.5% and 4.5%, respectively, and the volatilities of stock returns and the exchange rate are both 0.5. The other parameters are the same as in Table 1. In Figures 1 and 2, we show the required hedging positions of underlying stocks and foreign pure discount bonds (with par value USD 10,000) for hedging the fluctuations of stock price and the instantaneous correlation coefficient. From Figure 1, we can see that the higher the $\rho_{SQ}$, the larger the hedging position of underlying stock required. We can also observe that if stock price is close to the strike price, the required position for hedging changes rapidly. On the other hand, as the stock price moves away from the strike price, the
Table 1  The Valuation of High Yield Notes

This table reports the price of high yield notes as a function of the instantaneous correlation coefficient of stock and exchange rate $\rho_{SQ}$, annual volatility of the rate of return of underlying stock $\hat{\sigma}_S$, annual volatility of exchange rate $\hat{\sigma}_Q$, domestic risk-free rate $r_d$, and foreign risk-free rate $r_f$. The initial stock price is TWD 50. Time to maturity is 3 months. The spot exchange rate (TWD/USD) is 34.46. The strike price is TWD 42.5. Par value of the high yield note is USD 10,000. The following numerical results show that the value of the high yield note is an increasing function of the domestic risk-free interest rate, the instantaneous correlation coefficient of stock and the exchange rate and a decreasing function of the volatility of stock returns and the foreign risk-free interest rate. However, the sign of the instantaneous correlation coefficient determines the influence of the volatility of the exchange rate on the values of high yield notes. When $\rho_{SQ} > 0$, the value of the high yield note is increasing with the volatility of the exchange rate. If $\rho_{SQ} < 0$, the value of the high yield note is decreasing with an increase of exchange rate volatility.

<table>
<thead>
<tr>
<th>$\rho_{SQ}$</th>
<th>$\hat{\sigma}_S$</th>
<th>$\hat{\sigma}_Q$</th>
<th>$(r_d, r_f)$</th>
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<tr>
<td></td>
<td>(3.5%, 3.5%)</td>
<td>(3.5%, 4.5%)</td>
<td>(4.5%, 3.5%)</td>
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<tr>
<td>0.75</td>
<td>0.5</td>
<td>9,622.4585</td>
<td>9,598.4324</td>
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<td>0.2</td>
<td>9,563.1922</td>
<td>9,539.3140</td>
</tr>
<tr>
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<td>0.5</td>
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The required hedging volume of stock changes only mildly. Figure 2 shows the relationship of required volume of foreign pure discount bond (par value at USD 10,000) for hedging the fluctuations in the stock price and instantaneous correlation coefficient $\rho_{SQ}$. The higher the $\rho_{SQ}$, the higher the volume for hedging required. For the impact of stock price, when the stock price gradually approaches the strike price, the required volume for hedging increases rapidly. As the stock price rises above the strike price, the high yield notes will behave more like a foreign pure discount bond and hence the required volume of the foreign pure discount bond is close to one when the stock price is large. In sum, high yield notes provide the issuers an insurance on stock position; it is more useful, especially in a bear stock market when there are legal restrictions on issuing put options on equities.

7. PRICING HIGH YIELD NOTES UNDER THE GAUSSIAN HJM FRAMEWORK

Under the normal economic conditions, we can assume the short-term risk-free rate in a short period to be a constant. However, in some circumstances, the short-term interest rate changes dramatically. For example, when the central bank suddenly changes monetary policy or when oil shocks occur, the short-term interest rate will become volatile. Thus, instead of assuming constant interest rate, we now derive the pricing formulas under stochastic short-term rates. There are many interest rate models available, such as Vasicek (1977), Ho and Lee (1986), Cox et al. (1985), Hull and White (1993) etc. In this paper, we choose the forward interest rate model of HJM (1992) (which considers the dynamic process of instantaneous forward rates and provides general results that must hold for all arbitrage-free yield curve models) to value the price of high yield notes under stochastic interest rates.

We briefly review HJM (1992) and Amin and Jarrow (1991, 1992) setup and extend the HJM methodology to an international economy. Then, we extend the risk-neutral world in the previous section to the forward-neutral world developed by Geman (1989) and Jamshidian (1989).

Assumption 1 There are no transaction costs, taxes, and restrictions on short selling or other market imperfections in the security economy with a trading interval of $[0, T^*]$. 

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This figure shows the required volume of stock for hedging, under various stock prices and instantaneous correlation coefficients $\rho_{SQ}$. The higher the $\rho_{SQ}$, the larger the volume of underlying stock for hedging required. We can also observe that when the price is close to the strike price, the required volume for hedging changes rapidly. On the other hand, when the stock price is below the strike price, the required volume of stock changes only mildly.

**Figure 1** The Required Volume of Underlying Stock as Function of Stock Price and Correlation Coefficient $\rho_{SQ}$

Let $W$ be the $h$-dimensional standard Brownian motion given in a filtered probability space $(\Omega, F, P, (F_t)_{t=0}^{T^*})$. Define $f_i(t, T), 0 \leq t \leq T \leq T^*$, as the forward rate at time $t$ in the $i^{th}$ foreign market for instantaneous borrowing and lending at time $T$. We assume that $i = d$ represents a quantity related to the domestic market.

**Assumption 2** Given an initial forward rate curve $\{f_i(0, T) : T \in [0, T^*]\}$, the dynamics of the instantaneous forward rates in the $i^{th}$ foreign market are given by the integrated version of (15):
This figure shows the required volume of foreign pure discount bond (par value at USD 10,000) for hedging the changes in the stock price and correlation coefficient $\rho_{SQ}$. The higher the $\rho_{SQ}$, the higher the volume for hedging required. When the price is close to the strike price, the required volume for hedging increases rapidly. When the stock price is above the strike price, the high yield note will behave more like a foreign pure discount bond. Consequently, the required volume of foreign pure discount bond is close to one.

![Graph showing the required volume of foreign pure discount bond as a function of stock price and correlation coefficient $\rho_{SQ}$](image)

**Figure 2** The Required Volume of Foreign Pure Discount Bond as Function of Stock Price and Correlation Coefficient $\rho_{SQ}$

$$f_i(t, T) = f_i(0, T) + \int_0^t \alpha_i(u, T) du + \int_0^t \sigma_i(u, T) \cdot dW_t, \quad \forall t \in [0, T], \quad (15)$$

where $\alpha_i(t, T) : C \to R$, $\sigma_i(t, T) : C \to R^h$, and $C = \{(u, t) | 0 \leq u \leq t \leq T^*\}$. $\alpha_i(t, T)$ and $\sigma_i(t, T)$ are assumed to be adapted with respect to the filtration $F = F^W$. 

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and jointly measurable and uniformly bounded on \( \{ (t, v) : 0 \leq t \leq v \leq T^* \} \). Let \( B_i(t, T) \) be the time \( t \) price of a zero coupon bond paying one dollar at time \( T \) in the \( i^{th} \) foreign market. By definition of the forward rate (for details see HJM (1992)),

\[
\frac{dB_i(t, T)}{B_i(t, T)} = [r_i(t) + a_i(t, T)] dt + b_i(t, T) \cdot dW_t,
\]

where \( b_i(t, T) = -\int_t^T \sigma_i(t, v) dv, a_i(t, T) = -\int_t^T \alpha_i(t, v) dv + (1/2)|b_i(t, T)|^2 \).

We assume the dynamics of a primary security in the \( i^{th} \) foreign market, \( S_i \), are given by the following expression:

\[
\frac{dS_i(t)}{S_i(t)} = [u_i(t) - \delta_i(t)] dt + \sigma_{S_i}(t) \cdot dW_t,
\]

where \( \sigma_{S_i}(t) \in \mathbb{R}^h \) is a time-varying volatility. \( \delta_i(t) \) represents the dividend payout rate at time \( t \). \( u_i(t) \) and \( \delta_i(t) \) are \( F_t \)-adapted, jointly measurable and uniformly bounded in \( t \in [0, T^*] \). They also satisfy

\[
E \left[ \int_0^{T^*} [u_i(t) - \delta_i(t)]^2 dt \right] < \infty.
\]

The exchange rate process, \( Q_i(t) \), which is used to convert \( i^{th} \) foreign payouts into domestic currency, is modeled by the following stochastic differential equation:

\[
\frac{dQ_i(t)}{Q_i(t)} = u_{Q_i}(t) dt + \sigma_{Q_i}(t) \cdot dW_t.
\]

Let \( P_i^* \) be a risk-neutral probability measure in \( (\Omega, F) \), which uses the money market account as the numeraire. Consequently, we have Radon-Nikodym derivative

\[
\frac{dP_i^*}{dP} = \exp \left[ \int_0^{T^*} \eta_v^i dW_v - \frac{1}{2} \int_0^{T^*} |\eta_v^i|^2 dv \right].
\]

By Girsanov’s theorem, \( W^i \), defined below, is a \( P_i^* \)-Brownian motion:
\[ W_t^{i*} = W_t - \int_0^t \eta^i_v dv, \]

where \( \eta^i_t \in \mathbb{R}^h \) is the vector of market prices of risk corresponding to the sources of randomness in the economy and satisfies

\[ u_i(t) - \delta_i(t) - r_i(t) + \sigma S_i(t) \cdot \eta^i_t = 0, \]
\[ a_i(t, T) + b_i(t, T) \cdot \eta^i_t = 0, \]
\[ u_{Q_i}(t) + r_i(t) - r_d(t) + \sigma_{Q_i}(t) \cdot \eta^i_t = 0, \]

where \( r_i(t) \) equals the instantaneous spot rate at time \( t \), i.e., \( r_i(t) = f_i(t, t) \) in the \( i^{th} \) foreign market.

As a result, the dynamics of the prices of the zero coupon bond and the primary security under risk neutral probability measure \( P^{i*} \) are

\[ \frac{dB_i(t, T)}{B_i(t, T)} = r_i(t) dt + b_i(t, T) \cdot dW_t^{i*}, \]
\[ \frac{dS_i(t)}{S_i(t)} = [r_i(t) - \delta_i(t)] dt + \sigma S_i(t) \cdot dW_t^{i*}, \]

respectively. The dynamics of exchange rate process \( Q_i(t) \) under domestic risk neutral probability measure \( P^{d*} \) are

\[ \frac{dQ_i(t)}{Q_i(t)} = (r_d(t) - r_i(t)) dt + \sigma_{Q_i}(t) \cdot dW_t^{d*}. \]

Meanwhile, to exclude arbitrage opportunities between risk-free and risky investments in both economies, the probability measure \( P^{i*} \), equivalent to \( P^{d*} \), satisfies

\[ \frac{dP^{i*}}{dP^{d*}} = \exp \left[ \int_0^{T^*} \sigma_{Q_i}(v) dW_v^{d*} - \frac{1}{2} \int_0^{T^*} |\sigma_{Q_i}(v)|^2 dv \right]. \]

By Girsanov’s theorem, \( W^{i*} \), defined below, is a \( P^{i*} \)-Brownian motion:
The Valuation and Hedging Strategy of High Yield Notes (Szu-Lang Liao and Chou-Wen Wang)

\[ W_t^{i^*} = W_t^{d^*} - \int_0^t \sigma Q_i(v)dv. \]

Let \( F_S^i(t, T) \) be the forward price of a primary security \( S_i(t) \) after dividend payment at time \( t \), for settlement date \( T \). Then \( F_S^i(t, T) \) satisfies

\[ F_S^i(t, T) = \frac{S_i(t)}{B_i(t, T)}, \quad \forall t \in [0, T]. \]

Define a probability measure \( P_T^i \) on \((\Omega, F)\) equivalent to \( P^i\) with the Radon-Nikodym derivative given by

\[
\frac{dP_T^i}{dP^i} = \exp \left[ \int_0^T b_i(v, T) \cdot dW^i_T - \frac{1}{2} \int_0^T |b_i(v, T)|^2 dv \right].
\]

Define \( W_t^{i^T} \) as

\[ W_t^{i^T} = W_t^{i^*} - \int_0^t b_i(v, T) \cdot dv, \quad \forall t \in [0, T]. \]

Then \( W_t^{i^T} \) is an \( h \)-dimensional standard Brownian motion under the probability measure \( P_T^i \). \( P_T^i \) is called the forward neutral probability measure for the settlement time \( T \) in the \( i \)th foreign market. Hence, the forward price of a primary security \( S_i(t) \) satisfies

\[ F_S^i(t, T) = F_S^i(0, T) \exp \left[ \int_0^T \gamma_i(v, T) \cdot dW^{i^T}_v - \int_0^T \left( \delta_i(v) + \frac{1}{2} |\gamma_i(v, T)|^2 \right) dv \right], \]

where \( \gamma_i(t, T) = \sigma S_i(t) - b_i(t, T) \). Hence, under the Gaussian HJM framework, the valuation of high yield notes under domestic forward neutral probability measure \( P_T^d \) is as follows:

\[ C(t) = B_d(t, T) E_{P_T^d}[C(T)|F_t] \]

\[ = B_d(t, T) E_{P_T^d} \left[ M_f Q_f(T) I[S_d(T_D) \geq K] + \frac{M_f Q_f(T) S_d(T)}{K} \cdot I[S_d(T_D) < K]|F_t \right]. \quad (16) \]
We provide the closed-form solution of the high yield note under the Gaussian HJM framework in Theorem 3.

**Theorem 3**  The closed-form pricing formula of a high yield note under the Gaussian HJM framework is as follows:

\[
C(t) = MQ_f(t)B_f(t,T) \left\{ N(e_2) + \frac{S_d(t)}{KB_d(t,T)} \exp[G_3(t,T)] N(-e_1) \right\}, \tag{17}
\]

\[
e_1 = \frac{\ln \left[ \frac{S_d(t)}{KB_d(t,T_D)} \right] + G_1(t,T)}{v_S(t,T_D)},
\]

\[
e_2 = \frac{\ln \left[ \frac{S_d(t)}{KB_d(t,T_D)} \right] + G_2(t,T)}{v_S(t,T_D)},
\]

where

\[
G_1(t,T) = \int_t^{T_D} \{ \gamma_d(u,T_D) \cdot [0.5\gamma_d(u,T_D) + \sigma_Q(u,T)] - \delta_d(u) \} du,
\]

\[
G_2(t,T) = \int_t^{T_D} \{ \gamma_d(u,T_D) \cdot [0.5\gamma_d(u,T_D) - \gamma_d(u,T) + \sigma_Q(u,T)] - \delta_d(u) \} du,
\]

\[
G_3(t,T) = \int_t^T [\gamma_d(u,T) \cdot \sigma_Q(u,T) - \delta_d(u)] du,
\]

\[
v_S^2(t,T) = \int_t^T |\gamma_d(u,T)|^2 du,
\]

\[
\sigma_Q(t,T) = \sigma_{Q_f}(t) + b_f(t,T) - b_d(t,T).
\]

We prove Theorem 3 in Appendix.

Let \( \sigma_{S_d}(t) = \sigma_S, \sigma_{Q_d}(t) = \sigma_Q, b_i(t,T) = 0, \) and \( \delta_d(t) = \delta, \) the closed-form solution of (17) will be the same as (11) in Theorem 2. Furthermore, we may specify the domestic and foreign term structures in the same form generated by the following short-term rate processes:

\[
dr_d(t) = [\theta_d(t) - \beta_d(t)r_d(t)] dt + \sigma_d^2(t) \cdot dW_t^d, \tag{18}
\]

\[
dr_f(t) = [\theta_f(t) - \beta_f(t)r_f(t) + \rho_{DF} \sigma_Q(t)\sigma_f^2(t)] dt + \sigma_f^2(t) \cdot dW_t^d, \tag{19}
\]
where \( r_i(t) = f_i(t, t), \ i = d, f \) and \( \rho_{DF} \) is the correlation coefficient between the domestic spot interest rate and the foreign spot interest rate. The dynamics specified in (18) and (19) are known as an extended Vasicek model, which is a special case of the HJM framework (see Chiarella and Kwon (2001)); the closed-form solution of high yield notes under the extended Vasicek framework is the same as in Theorem 3 if the volatilities of domestic and foreign zero coupon bonds are set as

\[
b_i(t, T) = -C_i(t, T)\sigma_i(t), \quad \text{where} \quad C_i(t, T) = \int_t^T \exp\{-[b_i(s) - b_i(t)]\} ds \quad \text{and} \quad b_i(t) = \int_0^t \beta_i(u) du
\]

for \( i = d, f \). Hence, our pricing formula (17) derived under the HJM model of forward interest rates can be easily transformed to obtain the corresponding pricing formula under the extended Vasicek model of short-term interest rates.

### 8. Hedging Strategy Under the Gaussian HJM Framework

In this section, we provide the appropriate hedging strategies for the issuers under a stochastic interest rate environment. To hedge the high yield notes, we construct a hedged portfolio by short selling a high yield note and buying \( m^* B_d(t, T) \) units of the domestic underlying assets and \( n^* B_d(t, T) \) units of foreign pure discount bonds. The value of this portfolio \( W(t) \) is

\[
W(t) = -C(t) + m^* F^d_S(t, T) + n^* F^Q_S(t, T),
\]

where \( F^Q_S(t, T) = [Q_f(t) B_f(t, T) / B_d(t, T)] \). By Ito’s lemma we obtain

\[
dW(t) = -[C_H + m^* \delta_d(t)] dt + [(m^* - C^S_F) \gamma_d(t, T) + (n^* - C^S_Q) \sigma_Q(t, T) \cdot dW^d_T, \]

where

\[
C_H = -\delta_d(t) C^S_F + C_t + \frac{1}{2} [C^S_S F_S | \gamma_d(t, T) |^2 + 2 C^S_F Q \gamma_d(t, T) \cdot \sigma_Q(t, T) + C^Q_S Q | \sigma_Q(t, T) |^2.
\]
When $m^* = C_{FS}$ and $n^* = C_{FQ}$, the hedging portfolio is riskless. Hence, for hedging purposes, the issuers of high yield notes should buy $C_{FS}/B_d(t,T)$ units of the domestic underlying assets and $C_{FQ}/B_d(t,T)$ units of the foreign pure discount bonds. From Theorem 3 and the chain rules given in (12), we have

$$C_{FS} = MQ_f(t)B_f(t,T) \left\{ \frac{B_d(t,T)}{S_d(t)v_S(t,T_D)\sqrt{2\pi}} \exp\left[-\frac{e_2^2}{2}\right] - \frac{1}{Kv_S(t,T_D)\sqrt{2\pi}} \exp[G_3(t,T) - \frac{e_1^2}{2}] + \frac{N(-e_1)}{K} \exp[G_3(t,T)] \right\},$$

$$C_{FQ} = \frac{C(t)}{F_Q(t,T)}.$$

If $\sigma_{S_d}(t) = \sigma_S$, $\sigma_{Q_f}(t) = \sigma_Q$, $b_i(t,T) = 0$, and $\delta_{d}(t) = \delta$, the hedging positions are consistent with the constant interest rate case.

9. NUMERICAL ANALYSES OF HIGH YIELD NOTES UNDER STOCHASTIC INTEREST RATES

In this section, we investigate the properties of high yield notes numerically under the extended Vasicek model of short-term interest rates which is a special case of the HJM framework. Taking a one-year high yield note as an example, the par value is USD (U.S. dollar) 10,000 per note and the strike price is 85% of the reference spot price. The current exchange rate is 34.46. The determination date is two days before the maturity date. For simplicity, we assume that $\beta_d(t) = \beta_f(t) = 0.1$, and the spot interest rates for both countries are 1.65%, $\sigma_{S_d}(t) = 0.3$, $\sigma_{Q_f}(t) = 0.2$. Denote the relevant coefficients of correlation as follows:
where $\rho_{SQ}$ refers to the coefficient of correlation between foreign stock price and exchange rate, and other correlations have analogous definitions. We assume $\rho_{SQ} = 0.3$, $\rho_{DF} = 0.5$, $\rho_{FQ} = 0.5$, and summarize the numerical results in Table 2.

From Table 2, we see that the value of the high yield note is an increasing function of $\rho_{FS}$ and $\rho_{DS}$, and is a decreasing function of $\rho_{DQ}$. However, the signs of instantaneous correlation coefficients $\rho_{FS}$ and $\rho_{DS}$ determine the impacts of the volatilities of domestic and foreign interest rates. When $\rho_{FS} > 0$ ($\rho_{DS} > 0$), the value of the high yield note increases with the volatility of the foreign (domestic) interest rate. If $\rho_{FS} < 0$ ($\rho_{DS} < 0$), the value of the high yield note decreases with the increase of the volatility of foreign (domestic) interest rate.

## 10. CONCLUSION

In this paper, we provide the closed-form solution of the foreign equity options: exchange-linked stock options. We also provide the closed-form solutions, hedging strategies and some specific properties of the high yield notes under constant and stochastic interest rates. High yield notes provide the issuers, who have the stock positions, an insurance to hedge the price risk of stock positions caused by an economic downturn. This is especially useful under a bear stock market with legal restrictions on issuing put options on equities.

Using the framework developed in this paper, one can derive the closed-form valuation of various forms of equity-linked notes. One of them is the principal-protected notes which are also planned to be issued by Taiwanese local security dealers in the near future.
Table 2  The Valuation of High Yield Notes under Stochastic Interest Rates

This table reports the price of high yield notes as function of instantaneous correlation coefficients under an extended Vasicek model. We assume that $\beta_d(t) = \beta_f(t) = 0.1$, the spot interest rates for both countries are 1.65%, $\sigma_{S_d}(t) = 0.3$, $\sigma_{Q_f}(t) = 0.2$, $\rho_{SQ} = 0.3$, $\rho_{DF} = 0.5$, and $\rho_{FQ} = 0.5$. The spot exchange rate (TWD/USD) is 34.46. The strike price is 85% of the reference spot price. Par value of the high yield note is USD 10,000. The following numerical results show that the value of the high yield note is an increasing function of $\rho_{FS}$ and $\rho_{DS}$, and is a decreasing function of $\rho_{DQ}$. When $\rho_{FS} > 0$ ($\rho_{DS} > 0$), the value of the high yield note increases with the volatility of foreign (domestic) interest rate. If $\rho_{FS} < 0$ ($\rho_{DS} < 0$), the value of the high yield note decreases with the increase of the volatility of foreign (domestic) interest rate.

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APPENDIX

Proof of Theorem 1
We divide (6) into two parts:

\[ I_1 = -\exp[-r_d(T-t)]E_{P^d^*}[Q_f(T)S_d(T)I[K \geq S_d(T_D)]]|F_t], \]
\[ I_2 = K \exp[-r_d(T-t)]E_{P^d^*}[Q_f(T)I[K \geq S_d(T_D)]]|F_t]. \]

To compute \( I_1 \), we define \( S_d^*(t) = S_d(t)Q_f(t) \), then

\[ S_d^*(T) = S_d^*(t) \exp \left[ (2r_d - r_f - \delta + \rho_{SQ}\hat{\sigma}_S\hat{\sigma}_Q - \frac{1}{2}|\sigma_S + \sigma_Q|^2)(T-t) \right. \]
\[ \left. + (\sigma_S + \sigma_Q) \cdot (W^{d*}_T - W^{d*}_t) \right]. \] (A1)

Introducing an auxiliary probability measure \( P_R \) on \((\Omega, F)\), defined by the Radon-Nikodym derivative

\[ \frac{dP_R}{dP^{d*}} = \exp \left( (\sigma_S + \sigma_Q) \cdot W^{d*}_T - \frac{1}{2}|\sigma_S + \sigma_Q|^2T^* \right). \]

By Girsanov’s theorem, the process \( W^R_t \), defined by

\[ dW^R_t = dW^{d*}_t - (\sigma_S + \sigma_Q)dt \] (A2)

is a standard Brownian motion under probability measure \( P_R \). Hence,

\[ I_1 = -Q_f(t)S_d(t) \exp[(\rho_{SQ}\hat{\sigma}_S\hat{\sigma}_Q + r_d - r_f - \delta)(T-t)]E_{P_R}[I[K \geq S_d(T_D)]]|F_t] \] (A3)

and
Substituting (A4) into (A3), we obtain

\[ I_1 = -Q_f(t) S_d(t) \exp\left[\left(\rho_{SQ} \hat{\sigma}^S \hat{\sigma}^Q + r_d - r_f - \delta\right)(T - t)\right] E_{P_R} \left[ I \left| \ln \frac{S_d(T_D)}{K} \leq 0 \right| F_t \right] \]

\[ = -Q_f(t) S_d(t) \exp\left[\left(\rho_{SQ} \hat{\sigma}^S \hat{\sigma}^Q + r_d - r_f - \delta\right)(T - t)\right] \mathcal{N}(-d_1). \]

To compute \( I_2 \), we construct another probability measure \( P_Q \) on \((\Omega, \mathcal{F})\), defined by the Radon-Nikodym derivative

\[ \frac{dP_Q}{dP} = \exp \left( \sigma_Q \cdot W^*_t - \frac{1}{2} |\sigma_Q|^2 T^* \right). \]

By Girsanov’s theorem, the process \( W^Q_t \), defined by

\[ dW^Q_t = dW^*_t - \sigma_Q dt, \]

is a standard Brownian motion under probability measure \( P_Q \). Under the probability measure \( P_Q \), domestic stock \( S_d(t) \) satisfies

\[ S_d(T) = S_d(t) \exp \left[ \left(r_d - \delta + \rho_{SQ} \hat{\sigma}^S \hat{\sigma}^Q - \frac{1}{2} \hat{\sigma}^2_S\right)(T - t) + \sigma_S \cdot (W^R_T - W^R_t) \right]. \]

Then,

\[ I_2 = K Q_f(t) \exp[-r_f(T - t)] E_{P_Q} \left[ I \left| \ln \frac{S_d(T_D)}{K} \leq 0 \right| F_t \right] \]

\[ = K Q_f(t) \exp[-r_f(T - t)] \mathcal{N}(-d_2). \]

This completes the proof of (7).

To prove the put-call parity of domestic exchange-linked options, we know that

\[ \text{ELC}(T) = Q_f(T) \max[S_d(T) - K, 0]; \quad \text{ELP}(T) = Q_f(T) \max[K - S_d(T), 0]. \]
At time $t$, we buy one unit of exchange-linked put option, short sell one unit of exchange-linked put option and borrow a foreign pure discount bond with maturity date $T$. On the expiry date $T$, the payoff of this portfolio in units of domestic currency is $Q_f(T)S_d(T)$. Consequently, by (A1), the arbitrage-free value of this portfolio at time $t$ is

$$\exp[-r_d(T - t)]E_{P^r}[Q_f(T)S_d(T)|F_t] = Q_f(t)S_d(t)\exp[(\rho_{SQ}\hat{\sigma}_S\hat{\sigma}_Q + 2r_d - r_f - \delta) (T - t)].$$

This completes the proof of (8).

**Proof of Theorem 2**

We divide (10) into two parts:

$$I_1 = \exp[-r_d(T - t)]E_{P^r}\left[\frac{M_fQ_f(T)S_d(T)}{K}I[S_d(T_D) < K]|F_t\right],$$

$$I_2 = \exp[-r_d(T - t)]E_{P^r}[M_fQ_f(T)I[S_d(T_D) \geq K]|F_t].$$

To compute $I_1$, we define $S^*_d(t) = S_d(t)Q_f(t)$, then

$$S^*_d(T) = S^*_d(t)\exp\left[(2r_d - r_f - \delta + \rho_{SQ}\hat{\sigma}_S\hat{\sigma}_Q + \frac{1}{2}|\sigma_S - \sigma_Q|^2)(T - t)\right]$$

$$\quad + \sigma_S \cdot (W^*_T - W^*_t).$$

So,

$$I_1 = \frac{M_fQ_f(t)S_d(t)}{K}\exp[(\rho_{SQ}\hat{\sigma}_S\hat{\sigma}_Q + r_d - r_f - \delta)(T - t)] \cdot E_{P^r}[I[S_d(T_D) < K]|F_t],$$

(B1)

where the probability measure $P^r$ is the same as before. Under $P^r$, $S_d(t)$ satisfies
\[ S_d(T_D) = S_d(t) \exp \left[ \left( r_d - \delta + \rho_{SQ}\hat{\sigma}_S\hat{\sigma}_Q + \frac{1}{2}\hat{\sigma}_S^2 \right)(T_D - t) \right. \]
\[ + \sigma_S \cdot (W^R_{TD} - W^R_t) \right], \quad (B2) \]

where \( W^R_t \) is defined in (A2). Substituting (B2) into (B1), we obtain

\[ I_1 = \frac{M_f Q_f(t) S_d(t)}{K} \exp \left[ \left( \rho_{SQ}\hat{\sigma}_S\hat{\sigma}_Q + r_d - r_f - \delta \right)(T - t) \right] \]
\[ E_{P_R}[\ln S_d(T_D) < 0 | F_t] \]
\[ = \frac{M_f Q_f(t) S_d(t)}{K} \exp \left[ \left( \rho_{SQ}\hat{\sigma}_S\hat{\sigma}_Q + r_d - r_f - \delta \right)(T - t) \right] N(-d_1). \]

Similarly, for \( I_2 \), we use the probability measure \( P_Q \) and standard Brownian motion \( W^Q_t \) as defined earlier. Then

\[ I_2 = M_f Q_f(t) \exp[-r_f(T - t)] E_{P_Q}[\ln S_d(T_D) \geq K | F_t]. \tag{B3} \]

Under the probability measure \( P_Q \), domestic stock \( S_d(t) \) satisfies

\[ S_d(T_D) = S_d(t) \exp \left[ \left( r_d - \delta + \rho_{SQ}\hat{\sigma}_S\hat{\sigma}_Q - \frac{1}{2}\hat{\sigma}_S^2 \right)(T_D - t) \right. \]
\[ + \sigma_S \cdot (W^Q_{TD} - W^Q_t) \right]. \tag{B4} \]

Substituting (B4) into (B3), we obtain

\[ I_2 = M_f Q_f(t) \exp[-r_f(T - t)] P_Q \left[ \ln \frac{S_d(T_D)}{K} \geq 0 | F_t \right] \]
\[ = M_f Q_f(t) \exp[-r_f(T - t)] N(d_2). \]

This completes the proof of Theorem 2.
Proof of Theorem 3
We divide (16) into two parts:

\[ I_1 = B_d(t,T) E_{P^d} [M_f Q_f(T) I[S_d(T_D) \geq K] | F_t], \]
\[ I_2 = \frac{M_f}{K} B_d(t,T) E_{P^d} [Q_f(T) S_d(T) I[S(T_D) < K] | F_t]. \]

For \( I_1 \), we redefine \( F_Q(t,T) = Q_f(t) B_f(t,T) / B_d(t,T) \), by Ito’s lemma, we have

\[ \frac{dF_Q(t,T)}{F_Q(t,T)} = \sigma_Q(t,T) \cdot dW^d_T. \]

Then,

\[ I_1 = M_f Q_f(t) B_f(t,T) E_{P_X} [I[S_d(T_D) \geq K] | F_t], \] (C1)

where we assume that there exists a martingale \( P_X \) on \((\Omega, F)\) defined by the Radon-Nikodym derivative

\[ \frac{dP_X}{dP^d_T} = \exp \left( \int_0^T \sigma_Q(u,T) \cdot W^d_u dt - \frac{1}{2} \int_0^T |\sigma_Q(u,T)|^2 du \right). \]

Then, \( W^X_t \), defined by

\[ dW^X_t = dW^d_T - \sigma_Q(t,T) dt, \]

is a standard Brownian motion under probability measure \( P_X \). We know that the dynamics of \( F^d_S(t,T_D) \) under probability measure \( P^d_T \) are as follows:

\[ \frac{dF^d_S(t,T_D)}{F^d_S(t,T_D)} = \{ \gamma_d(t,T_D) \cdot [\gamma_d(t,T_D) - \gamma_d(t,T)] - \delta_d(t) \} dt + \gamma_d(t,T_D) \cdot dW^d_T. \]

Then, under probability measure \( P_X \), the dynamics of domestic asset \( S_d(T_i) \) satisfy
\[ S_d(T_D) = \frac{S_d(t)}{B_d(t, T_D)} \cdot \exp \left[ \int_t^{T_D} \gamma_d(u, T_D) \cdot dW_u^X + G_2(t, T) \right]. \quad (C2) \]

Substituting (C2) into (C1), we have

\[ I_1 = M f Q_f(t) B_f(t, T) \exp \left[ \ln \frac{S_d(T_D)}{K} \geq 0 | F_t \right] = M f Q_f(t) B_f(t, T) N(e_2). \]

For \( I_2 \), we define \( \tilde{F}_S^d(t, T) = F_S^d(t, T) F_Q(t, T) \), by Ito’s lemma we obtain

\[ \frac{d\tilde{F}_S^d(t, T)}{F_S^d(t, T)} = [\gamma_d(t, T) \cdot \sigma(t, T) - \delta_d(t)] dt + [\gamma_d(t, T) + \sigma(t, T)] \cdot dW^d_t. \]

Then,

\[ I_2 = \frac{M f Q_f(t) S_d(t) B_f(t, T)}{K B_d(t, T)} \exp[G_3(t, T)] exp_{P_Z} \left[ I[S_d(T_D) < K] | F_t \right], \quad (C3) \]

where we use another probability measure \( P_Z \) on \( (\Omega, \mathcal{F}) \), defined by the Radon-Nikodym derivative

\[ \frac{dP_Z}{dP} = \exp \left( \int_0^t [\gamma_d(u, T) + \sigma(u, T)] \cdot W^d_T - \frac{1}{2} \int_0^t |\gamma_d(u, T) + \sigma(u, T)|^2 du \right). \]

Then, \( W^Z_t \), defined by

\[ dW^Z_t = dW^d_t - [\gamma_d(t, T) + \sigma(t, T)] dt, \]

is a standard Brownian motion under probability measure \( P_Z \). Consequently, we have

\[ S_d(T_D) = \frac{S_d(t)}{B_d(t, T)} \cdot \exp \left[ \int_t^{T_D} \gamma_d(u, T_D) \cdot dW^Z_u + G_1(t, T) \right]. \quad (C4) \]

Substituting (C4) into (C3), we have
\[ I_2 = \frac{M_f Q_f(t) S_d(t) B_f(t, T)}{K B_d(t, T)} \exp[G_3(t, T)] E_{P_2} \left[ I \left[ \ln \frac{S_d(T_D)}{K} < 0 \right] F_t \right] \]

\[ = \frac{M_f Q_f(t) S_d(t) B_f(t, T)}{K B_d(t, T)} \exp[G_3(t, T)] N(-e_1). \]

This completes the proof of Theorem 3.
REFERENCES


高收益票券之訂價及避險策略

廖四郎 *
國立政治大學金融學系

王昭文
國立高雄第一科技大學財務管理學系

關鍵詞: 高收益票券、匯率連結選擇權、隨機利率、避險策略
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* 聯繫作者: 廖四郎, 國立政治大學金融學系, 台北市 116 文山區指南路二段 64 號。 電話: (02)
2939-3091 分機 81251; 傳真: (02) 2939-8004; E-mail: liaosl@nccu.edu.tw。
摘 要

本文導出台灣證券商將於 2003 年發行的高收益票券訂價公式的封閉解。這種金融衍生商品是零息債券加上一個短部位的股票賣權，其為證券商在面臨法律限制發行股票賣權或股市不景氣時的避險工具。本文證明高收益票券可以買入零息債券及賣出匯率連結股票賣權來複製，並且探討這些票券的特性及避險策略。最後，本文導出高收益票券在 HJM 架構之隨機利率下的訂價公式。