Sequentially Stable Coalition Structures*

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Abstract

In this paper, we examine the question of which coalition structures are formed in cooperative games with externalities. We introduce a stability concept for a coalition structure called a sequentially stable coalition structure by extending the concept of an equilibrium binding agreement (EBA) due to Ray and Vohra (1997). In an EBA, coalitions can only break up into smaller sizes of coalitions, but not merge into larger sizes of coalitions. On the other hand, in our concept of sequential stability, both breaking up and merging are allowed for coalitions. We also use a “step-by-step” approach to describe negotiation steps concretely by restricting how coalition structures can change: when one coalition structure is changed to another one, either (i) only one merging of two separate coalitions into a coalition occurs, or (ii) only one breaking up of a coalition into two separate coalitions happens. As an application of our stability notion, we show that the coalition structure consisting of only the grand coalition structure can be sequentially stable in common pool resource games.

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1 Introduction

In this paper, we examine the question of which coalition structures are formed in cooperative games with externalities. We introduce a stability concept for a coalition structure called a \emph{sequentially stable} coalition structure by extending the concept of an \emph{equilibrium binding agreement} (EBA) due to Ray and Vohra (1997). Ray and Vohra capture explicitly credibility of blocking coalitions, and then induce a recursive definition of a stable coalition structure. In their definition, however, coalitions can only break up into smaller sizes of coalitions, but cannot merge into larger sizes of coalitions. In particular, this means that the \emph{singleton} coalition structure consisting only of one-person coalitions is always an EBA.

On the other hand, in our concept of sequential stability, both breaking up and merging are allowed for coalitions. Moreover, we use a "step-by-step" approach to describe negotiation steps concretely by restricting how coalition structures can change: when one coalition structure is changed to another one, either (i) only one merging of two separate coalitions into a coalition occurs, or (ii) only one breaking up of a coalition into two separate coalitions happens. More specifically, our definition of domination between two coalition structures is given as follows. The coalition structure $\mathcal{P}$ is said to \emph{sequentially dominate} the coalition structure $\mathcal{P}'$ if there is a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^T$ starting from $\mathcal{P}_0 = \mathcal{P}$ to $\mathcal{P}_T = \mathcal{P}'$ such that at each step $t$, one of the following holds:

1. Two separate coalitions in $\mathcal{P}_t$ merge into one coalition in $\mathcal{P}_{t+1}$ and no other change occurs. All members in the two merging coalitions prefer the payoffs under the final coalition structure $\mathcal{P}'$ to those under the coalition structure $\mathcal{P}_t$ before merging; or
2. One coalition in $\mathcal{P}_t$ breaks up into two separate coalitions in $\mathcal{P}_{t+1}$ and no other change occurs. All members in at least one breaking up coalitions prefer the payoffs under the final coalition structure $\mathcal{P}'$ to those under the coalition structure $\mathcal{P}_t$ before breaking up.

A sequentially stable coalition structure is defined as a coalition structure which sequentially dominates all other coalition structures.

Diamantoudi and Xue (2002) recently propose another extension of the EBA notion called an extended EBA (EEBA). Breaking up as well as merging are possible for coalitions in their stability concept, too. However, both breaking up and merging could occur at the same time, and so a lot of possibilities of changes in coalition structures are allowed. For example, the singleton coalition structure in which no cooperation among players is formed can be suddenly changed into the \emph{grand} coalition structure in which all players cooperate. Then the following question naturally arises: through which negotiation steps is the singleton coalition structure changed to the grand coalition structure? More generally, how is some coalition structure transformed into another coalition structure when the two coalition structures are quite different? Our notion of sequential stability using a step-by-step approach to describe negotiation steps will give a more satisfactory answer to these questions than the EEBA concept does.

We compare the three stability notions, EBA's, EEBA's, and sequential stability. First, all of them are characterized by the von Neumann-Morgenstern stable sets with
respect to different domination relations. Second, sequential stability is a refinement of the EEBA notion in the sense that if a coalition structure $\mathcal{P}$ is sequentially stable, then the singleton set consisting only of $\mathcal{P}$ is an EEBA. However, the converse is not true: the singleton set consisting of one coalition structure that is not sequentially stable may be an EEBA. Moreover, there is no logical relation between sequential stability and the EBA notion.

We also identify a simple condition for which the grand coalition structure is a unique sequentially stable coalition structure in a partition function form game. Furthermore, as another application of our stability concept, we study a model of an economy with a common pool resource which has been often examined (e.g., Weitzman (1974), Roemer (1989), and Funaki and Yamato (1999)). We show that the grand coalition structure can be sequentially stable in common pool resource games.

The rest of the paper is organized as follows. In Section 2, we introduce notation and definitions. In Section 3, we define sequential stability of coalition structures and compare our notion with EBA’s and EEBA’s. In Section 4, as an application, we examine sequential stability in common pool resource games. Section 5 contains some concluding remarks.

2 Domination and Some Basic Concepts

Let $N = \{1, 2, \ldots, n\}$ be a set of players. A subset $S$ of $N$ is called a coalition. We use the concept of a coalition structure to express how players form coalitions. Here a coalition structure $\mathcal{P}$ is a partition $\{S_1, S_2, \ldots, S_k\}$ of $N$, where $S_1, S_2, \ldots, S_k$ in $\mathcal{P}$ are disjoint and $\bigcup_{j=1}^{k} S_j = N$. The set of partitions of $N$ is denoted by $\Pi(N)$.

We assume that given any coalition structure $\mathcal{P} \in \Pi(N)$, the feasible payoff vector under $\mathcal{P}$, $u(\mathcal{P}) = (u_1(\mathcal{P}), u_2(\mathcal{P}), \ldots, u_n(\mathcal{P})) \in \mathbb{R}^n$, is uniquely determined. The triple $(N, \Pi(N), (u_i)_{i \in N})$ is called a game with externalities.

We give two examples of games with externalities.

**Example 1.** Games in partition function form.

A game in partition function form $(N, v)$ is defined by a pair of a set of players $N$ and a partition function $v$ which assigns to each pair of a partition $\mathcal{P} \in \Pi(N)$ and a coalition $S \in \mathcal{P}$, a real number $v(S|\mathcal{P})$. Given a game in partition function form, the feasible payoff vector under $\mathcal{P}$ is given by $u_i(\mathcal{P}) = \frac{v(S|\mathcal{P})}{|S|} \forall i \in S, \forall S \in \mathcal{P}$. (See Thrall and Lucas(1963)).

**Example 2.** Hedonic games.

A hedonic game $(N, \{\succ_i\}_{i \in N})$ is defined by a pair of a set of players $N$ and a binary relation $\succ_i$ on $\{S \subset N | S \ni i\}$ for all $i \in N$, which represents $i$’s preference over coalitions that contain $i$. Consider $i$’s utility function $u_i$ over $\Pi(N)$ defined from $\succ_i$: For $\mathcal{P}$ and $\mathcal{P}’ \in \Pi(N)$, we define

$$u_i(\mathcal{P}) > u_i(\mathcal{P}’) \iff S \succ_i T,$$

where $i \in S, S \in \mathcal{P}$ and $i \in T, T \in \mathcal{P}’$. Then $(N, \Pi(N), (u_i)_{i \in N})$ becomes a game with externalities. (See, for example, Dreze and Greenberg (1980), Bogomalnia and Jackson(2002), Diamantoudi and Xue (2003).)
We introduce two special types of coalition structures. \( \mathcal{P}^N = \{N\} \) is called a \textit{grand} coalition structure, and \( \mathcal{P}^I = \{\{1\}, \{2\}, \ldots, \{n\}\} \) is called a \textit{singleton} coalition structure. We also say that \( \mathcal{P}' \) is a \textit{finer} coalition structure of \( \mathcal{P} \) (\( \mathcal{P} \) is a \textit{coarser} coalition structure of \( \mathcal{P}' \)) if the coalition structure \( \mathcal{P}' \) is given by re-dividing the coalition structure \( \mathcal{P} \), that is, \( \forall S' \in \mathcal{P}', \exists S \in \mathcal{P} \) such that \( S' \subseteq S \) and \( |\mathcal{P}'| > |\mathcal{P}| \).

We introduce several stability concepts for a set of coalition structures. This is an alternative way to define a core of a game with externalities. For this purpose, we define two simple concepts of dominations between two coalition structures.

**Definition 1.** Let \( \mathcal{P}, \mathcal{P}' \in \Pi(N) \). We say that \( \mathcal{P} \) dominates \( \mathcal{P}' \) if
1. \( \mathcal{P} \) is a finer coalition structure of \( \mathcal{P}' \), and
2. there exists \( T \in \mathcal{P} \) such that \( T \notin \mathcal{P}' \) and \( u_i(\mathcal{P}) > u_i(\mathcal{P}') \) \( \forall i \in T \).

**Definition 2.** Let \( \mathcal{P}, \mathcal{P}' \in \Pi(N) \). We say that \( \mathcal{P} \) directly dominates \( \mathcal{P}' \) if
1. \( \mathcal{P} \) is a finer coalition structure of \( \mathcal{P}' \), and
2. \( |\mathcal{P}| = |\mathcal{P}'| + 1 \),
3. there exists \( T \in \mathcal{P} \) such that \( T \notin \mathcal{P}' \) and \( u_i(\mathcal{P}) > u_i(\mathcal{P}') \) \( \forall i \in T \).

We define credible coalition structures by these definitions of dominations.

The following definition is a natural extension of Ray(1989)’s credible core for TU games to games with externalities.

**Definition 3.**
(1) \( \mathcal{P}^I = \{\{1\}, \{2\}, \ldots, \{n\}\} \) is \textit{credible}; and
(2) for \( k \) \( (k = n - 1, n - 2, \ldots, 1) \), \( \mathcal{P} \) with \( |\mathcal{P}| = k \) is \textit{credible} if \( \mathcal{P} \) is not directly dominated by any coalition structure \( \mathcal{P}' \) where \( \mathcal{P}' \) is credible and \( |\mathcal{P}'| = k + 1 \).

The set of all credible coalition structures is called the \textit{credible core}, and is denoted by \( CC \).

This is a recursive definition. First, according to (1), \( \mathcal{P}^I \) is credible. Second, we can check whether or not each of a coalition structure of \( (n - 1) \) coalitions is credible by using the fact \( \mathcal{P}^I \) is credible. Third, we can check whether or not each of a coalition structure of \( (n - 2) \) coalitions is credible by using the fact obtained in the second step, and so on.

Ray and Vohra (1997) extends the concept of the credible core in a different manner. Their concept is called an “equilibrium binding agreement (EBA)”. We express EBA’s in the following simple way using a recursive definition although this expression is different from the original one.

**Definition 4.**
(1) \( \mathcal{P}^I = \{\{1\}, \{2\}, \ldots, \{n\}\} \) is an \textit{EBA}; and
(2) for \( k \) \( (k = n - 1, n - 2, \ldots, 1) \), \( \mathcal{P} \) with \( |\mathcal{P}| = k \) is an \textit{EBA} if \( \mathcal{P} \) is not dominated by any coalition structure \( \mathcal{P}' \) where \( \mathcal{P}' \) is an EBA and \( |\mathcal{P}'| > k \).

The set of all EBA coalition structures is called the \textit{EBA core}.

The difference between the credible core and the EBA core is as follows: In a credible coalition structure, only the direct domination is considered, but in an EBA coalition structure, every possible domination is considered.
The following example shows that the concept of a credible coalition structure is different from an EBA. It is more difficult to find an EBA.

**Example 3.** Consider a symmetric 5-person game in partition function form \((N, v)\), where \(N = \{1, 2, 3, 4, 5\}\) and

\[
v(N|N) = 50. \quad \text{For any } P_2 \text{ s.t. } |P_2| = 2 \text{ and for any } S \in P_2, v(S|P_2) = 18.
\]

For any \(P_3 \text{ s.t. } |P_3| = 3\) and for any \(S \in P_3, v(S|P_3) = 8\).

For any \(P_4 \text{ s.t. } |P_4| = 4\) and for any \(S \in P_4, v(S|P_4) = 5\).

For any \(\{i\} \in P_I, v(\{i\}|P_I) = 3\).

Figure 1 shows that all possible coalition structures and the feasible payoff vectors under each coalition structure. Here the circle shows the coalition and the number in the circle indicates the cardinality of the coalition. The vector under each coalition shows the feasible payoffs \(u_i(P) = \frac{v(S|P)}{|S|}\). In this figure, every coalition structure \(P\) except for \(P_I\) is directly dominated by a coalition structure \(P'\) such that \(|P'| = |P| + 1\) (see Definition 2). Then it is easy to check that \(P_I^{I} = \{1; 1; 1; 1; 1\}, P^N = \{5\}, \{1; 2; 2\}, \{1; 1; 3\}\) are credible coalition structures (see Definition 3). On the other hand, it is not hard to see that \(\{1; 4\}, \{1; 2; 2\},\) and \(P_I\) are EBA’s (see Definition 4).

In both definitions of the credible core and EBA core, coalitions can only break up into smaller sizes of coalitions, but not merge into larger sizes of coalitions. In particular, this means that the singleton coalition structure consisting only of one-person coalitions belongs to both the credible and EBA cores. In the next section, we propose another new stability concept of coalition structures such that both breaking up and merging are allowed for coalitions.

### 3 Sequentially Stable Coalition Structures

In this section, we give our main stability concept called a “sequentially stable coalition structure”. First we give a definition of sequential domination, and after that we give a definition of a sequentially stable coalition structure.

**Definition 5.** Let \(P, P' \in \Pi(N)\). We say that \(P\) sequentially dominates \(P'\) if there is a sequence of coalition structures \(\{P_t\}_{t=0}^T\) such that

1. \(P_T = P\) and \(P_0 = P'\),
2. for all \(t\) \((0 \leq t \leq T - 1)\), either \(P_{t+1}\) is a finer coalition structure of \(P_t\) with \(|P_{t+1}| = |P_t| + 1\), or \(P_{t+1}\) is a coarser coalition structure of \(P_t\) with \(|P_{t+1}| = |P_t| - 1\), and
3. for all \(t\) \((0 \leq t \leq T - 1)\), for some \(S \in P_{t+1}\) with \(S \notin P_t\),

\[
u_i(P_t) < u_i(P_T) \quad \forall i \in S.
\]

\(^1\)Here \(\{1; 2; 2\}\) means every considerable coalition structure with one singleton and two 2-person coalitions and \(\{1; 1; 3\}\) means every considerable coalition structure with two singletons and one 3-person coalition.
We use the following notation for this sequence of coalition structures:

\[ P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \ldots \rightarrow P_T. \]

The condition (3) means that if \( P_{t+1} \) is a finer coalition structure of \( P_t \), for any member \( i \) in one of the two divided coalitions \( S \) and \( T \) such that \( S, T \in P_{t+1} \) and \( S \cup T \in P_t \), his payoff \( u_i(P_t) \) is smaller than his terminal payoff \( u_i(P_T) \); and if \( P_{t+1} \) is a coarser coalition structure of \( P_t \), for any member \( i \) in two combined coalitions \( S \) and \( T \) such that \( S, T \in P_t \) and \( S \cup T \in P_{t+1} \), his payoff \( u_i(P_t) \) is smaller than his terminal payoff \( u_i(P_T) \).

**Definition 6.** We say that \( P^* \in \Pi(N) \) is a sequentially stable coalition structure if for all other coalition structures \( P \neq P^* \), \( P^* \) sequentially dominates \( P \).

We will compare our domination relation with those of Ray and Vohra(1997) and Diamantoudi and Xue (2002). We have domination due to Ray and Vohra called RV-domination by changing the condition (2) in Definition 5 into the following condition (2').

**Definition 7.** Let \( P, P' \in \Pi(N) \). We say that \( P \) RV-dominates \( P' \) if there is a sequence of coalition structures \( \{P_t\}_{t=0}^T \) such that

1. \( P_T = P \) and \( P_0 = P' \),
2. \( P_t \) is a finer coalition structure of \( P_{t+1} \) with \( |P_{t+1}| = |P_t| + 1 \).
3. For all \( t \) (0 ≤ t ≤ T − 1), for some \( S \in P_{t+1} \) with \( S \notin P_t \), 
   \[ u_i(P_t) < u_i(P_T) \quad \forall i \in S. \]

Note in condition (2'), only refinement of coalition structures is allowed. The set of EBA coalition structures is defined by the following set \( \mathcal{E} \) of coalition structures such that

(a) \( P' \in \mathcal{E} \) and
(b) for any coalition structure \( P' \notin \mathcal{E} \), there exists \( P \in \mathcal{E} \) such that \( P \) RV-dominates \( P' \), and
(c) for any coalition structure \( P' \in \mathcal{E} \), there is no \( P \in \mathcal{E} \) such that \( P \) RV-dominates \( P' \).

Indeed the set \( \mathcal{E} \) is the vNM-stable set via RV-domination (Diamantoudi and Xue (2002)) because condition (b) corresponds to external stability of the vNM-stable set, and condition (c) corresponds to internal stability of the vNM-stable set. For our notion of sequential domination, the singleton set consisting of any sequentially stable coalition structure is also the vNM-stable set via that domination.

If we change the conditions (2) and (3) in Definition 5 into the following conditions (2'') and (3'), then we have a domination relation of Diamantoudi and Xue (2002) called DX-domination.
Definition 8. Let \( \mathcal{P}, \mathcal{P}' \in \Pi(N) \). We say that \( \mathcal{P} \) DX-dominates \( \mathcal{P}' \) if there is a sequence of coalition structures \( \{\mathcal{P}_t\}_{t=0}^T \) such that

1. \( \mathcal{P}_T = \mathcal{P}, \mathcal{P}_0 = \mathcal{P}' \), and

2. for all \( t (0 \leq t \leq T - 1) \), \( \mathcal{P}_{t+1} \) and \( \mathcal{P}_t \equiv \{S_1, S_2, \ldots, S_k\} \) satisfy the following condition; there exists a coalition \( Q(t) \subseteq N \) such that

   i. \( Q(t) = Q_1 \cup Q_2 \cup \ldots \cup Q_l \) where \( Q_j \in \mathcal{P}_{t+1} \) for all \( j = 1, 2, \ldots, l \) and \( Q_j \)s are disjoint,

   ii. \( \forall j = 1, 2, \ldots, k, S_j \cap Q(t) \neq \emptyset \Rightarrow S_j \setminus Q(t) \in \mathcal{P}_{t+1} \),

   iii. \( \forall j = 1, 2, \ldots, k, S_j \cap Q(t) = \emptyset \Rightarrow S_j \in \mathcal{P}_{t+1} \).

3. for all \( t (0 \leq t \leq T - 1) \),

   \( u_i(\mathcal{P}_t) < u_i(\mathcal{P}_T) \) \( \forall i \in Q(t) \).

The element of the vNM-stable set of the coalition structures using DX-domination is called the set of Extended EBA (EEBA) coalition structures. Condition (2") in Definition 8 implies that many possibilities of refining and merging are allowed for coalitions. On the other hand, the way of changing coalitions should be step by step and no jump are allowed in Definition 5 of sequential domination. For two coalition structures \( \mathcal{P} \) and \( \mathcal{P}' \), \( \mathcal{P} \) DX-dominates \( \mathcal{P}' \) if \( \mathcal{P} \) sequentially dominates \( \mathcal{P}' \) because (2) in Definition 5 implies (2") in Definition 8. Hence, sequential stability is a refinement of the notion of EEBA’s in the sense that if a coalition structure \( \mathcal{P} \) is sequentially stable, then the singleton set consisting only of \( \mathcal{P} \) is an EEBA. However, the converse is not true: the singleton set consisting of one coalition structure that is not sequentially stable may be an EEBA. Moreover, there is no logical relation between sequential stability and the notion of EBA’s. The following example illustrate these facts:

Example 4. Let us consider the 5-person game in Example 3 once more. We will show that the grand coalition structure \( \mathcal{P}^N \) is sequentially stable.

The proof consists of 5 steps.

1. \( \{2; 3\} \rightarrow \mathcal{P}^N \) because every player in \( \{2; 3\} \) gets more payoff at \( \mathcal{P}^N \).

2. \( \{1; 2; 2\} \rightarrow \{2; 3\} \rightarrow \mathcal{P}^N \) and \( \{1; 1; 3\} \rightarrow \{2; 3\} \rightarrow \mathcal{P}^N \) because every player in \( \{1; 2; 2\} \) and \( \{1; 1; 3\} \) gets more payoff at \( \mathcal{P}^N \) and (1) holds.

3. \( \{1; 1; 1; 2\} \rightarrow \{1; 2; 2\} \rightarrow \{2; 3\} \rightarrow \mathcal{P}^N \) because every player in \( \{1; 1; 1; 2\} \) gets more payoff at \( \mathcal{P}^N \) and (2) holds.

4. \( \{1; 1; 1; 1; 1\} \rightarrow \{1; 1; 1; 2\} \rightarrow \{1; 2; 2\} \rightarrow \{2; 3\} \rightarrow \mathcal{P}^N \) because every player in \( \{1; 1; 1; 1; 1\} \) gets more payoff at \( \mathcal{P}^N \) and (3) holds.

5. \( \{1; 4\} \rightarrow \{1; 1; 3\} \rightarrow \{2; 3\} \rightarrow \mathcal{P}^N \) because the deviation of one person in the 4-person coalition in \( \{1; 4\} \) increases his payoff at the final coalition structure \( \mathcal{P}^N \) and (2) holds.

These observations imply that the grand coalition structure \( \mathcal{P}^N \) is sequentially stable.

Moreover, \( \mathcal{P}^N \) is only one sequentially stable coalition structure. The reason is as follows: All the coalition structures except for \( \mathcal{P}^N \) and \( \{1; 4\} \) are not sequentially
stable because they are Pareto dominated by \( P^N \) and they cannot sequentially dominate \( P^N \). Next consider \{1; 4\}. Though the coalition structure \{1; 4\} is also Pareto efficient, it is not sequentially stable, because \{1; 4\} cannot sequentially dominate \{1; 1; 1; 2\}. 

On the other hand, \{1; 4\} DX-dominates \{1; 1; 1; 2\}. Hence it is easy to check that \{1; 4\} as well as \( P^N \) is an EEBA. Besides, as mentioned in Example 3, \{1; 4\}, \{1; 2; 2\}, and \( P^I \) are EBA’s.

The properties of EEBA’s are examined in Diamantoudi and Xue (2002). In their paper, they give the following proposition:

**Definition 9.** The coalition structure \( P \in \Pi(N) \) is Pareto efficient if there does not exist \( P' \in \Pi(N) \) such that \( u_i(P') > u_i(P) \) for any \( i \in N \).

**Proposition 1 (Diamantoudi and Xue (2002)).** Let \( P^* \in \Pi(N) \) be Pareto efficient. \( P^* \) is an EEBA if

(a) \( u_i(P^*) > u_i(P^I) \) \( \forall i \in N \), and

(b) for all \( P \in \Pi(N) \) such that \( P \neq P^* \) and \( P \neq P^I \), there is a coalition \( S \in P \) such that \( |S| > 1 \) and \( u_i(P^*) > u_i(P) \) for some \( i \in S \).

The similar proposition holds for our notion of sequential stability.

**Proposition 2.** Let \( P^* \in \Pi(N) \) be Pareto efficient. \( P^* \) is sequentially stable if

(a) \( P^* \) sequentially dominates \( P^I \), and

(b) for all \( P \in \Pi(N) \) such that \( P \neq P^* \) and \( P \neq P^I \), there is a coalition \( S \in P \) such that \( |S| > 1 \) and \( u_i(P^*) > u_i(P) \) for some \( i \in S \).

**Proof.** Take any \( P \) such that \( P \neq P^* \). We have to find a sequence of coalition structures from any \( P \) to \( P^* \) satisfying (1), (2) and (3) in Definition 6. First we construct a sequence \( \{P_k\}_{k=0}^R \) of coalition structures from \( P \) to \( P^I \), where \( P_0 = P \) to \( P_R = P^I \) \( (R \leq n) \). In the sequence \( \{P_k\}_{k=0}^R \), for any \( P_k \) such that \( P_k \neq P^I \), one person deviates from one of the largest coalition in \( P_k \). In this step, the deviated person prefers \( P^* \) to \( P \) because of (b). Second, (a) implies that the existence of a sequence of coalition structures from \( P^I \) to \( P^* \). Combining these sequences, we obtain the desired sequence of coalition structures. This implies \( P^* \) sequentially dominates \( P \).

Q.E.D.

We will give a simple condition for which only the grand coalition structure is sequentially stable in a partition function form game.

**Proposition 3.** Consider an \( n \)-person partition function form game which satisfies

\[
\frac{v(N|P^N)}{n} > \frac{v(S|P)}{|S|} \quad \forall P \in \Pi(N) \quad \forall S \in P.
\]

Then only \( P^N \) is sequentially stable.
Proof. Take any $\mathcal{P} \neq \mathcal{P}^N$. For any $S$ and $\mathcal{P}$ such that $S \in \mathcal{P}$, we have $u_i(\mathcal{P}^N) = \frac{v(S \cap \mathcal{P}^N)}{n} > \frac{v(S \cap \mathcal{P})}{|S|} = u_i(\mathcal{P})$ for all $i \in S$. Then first every member of the coalitions in $\mathcal{P}_2$ with 2 coalitions prefers the grand coalition structure, that is, $\mathcal{P}^N$ sequentially dominates $\mathcal{P}_2$. Similarly we can show that $\mathcal{P}^N$ sequentially dominates $\mathcal{P}_3, \mathcal{P}_4, \ldots, \mathcal{P}_I$, where $|\mathcal{P}_k| = k$. This shows that $\mathcal{P}^N$ is sequentially stable. It is obvious that $\mathcal{P}^N$ cannot be sequentially dominated by any other coalition structure.

Q.E.D.

The above result says that in a partition function form game, if the per capita value of the grand coalition is larger than that of any other coalition under any coalition structure, then the set of sequential stable coalition structures consists only of the grand coalition. Moreover, it coincides with the set of EEBA’s. However it is different from the EBA core because the EBA core contains $\mathcal{P}_I$ also.

4 Applications to Common Pool Resource Games

4.1 The Basic Model

We will apply the above concepts of the credible core, EBA’s, EEBA’s, and sequential stability to the following game of an economy with a common pool resource. For any player $i \in N$, let $x_i \geq 0$ represent the amount of labor input of $i$. Clearly, the overall amount of labor is given by $\sum_{j \in N} x_j$. The technology that determines the amount of product is considered to be a joint production function of the overall amount of labor $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $f(0) = 0$, $\lim_{x \to \infty} f'(x) = 0$, $f'(x) > 0$ and $f''(x) < 0$ for $x > 0$. The distribution of the product is supposed to be proportional to the amount of labor expended by players. In other words, the amount of the product assigned to player $i$ is given by $\frac{x_i}{\sum_{j \in N} x_j} \cdot f(\sum_{j \in N} x_j)$. The price of the product is normalized to be one unit of money and let $q$ be a cost of labor per unit, and we suppose $0 < q < f'(0)$.

Then individual $i$’s income is denoted by

$$m_i(x_1, x_2, \ldots, x_n) = \frac{x_i}{x_N} f(x_N) - qx_i.$$

The total income of coalition $S$ is denoted by

$$m_S \equiv \sum_{i \in S} m_i = \frac{x_S}{x_N} f(x_N) - qx_S,$$

where $x_S \equiv \sum_{i \in S} x_i$. We consider a game where each coalition is a player. It chooses its total labor input and its payoff is given by the sum of the income over its members. Naturally we can define a Nash equilibrium of that game.

Definition 10. The list $(x^*_1, x^*_2, \ldots, x^*_k)$ is an equilibrium under $\mathcal{P}$ if

$$m_{S_j}(x^*_{S_j}, x^*_{S_{-j}}) \geq m_{S_j}(x_{S_j}, x^*_{S_{-j}}), \quad \forall j, \quad \forall x_{S_j} \in \mathbb{R}_+.$$

There is a unique equilibrium under every coalition structure:
Proposition 4 (Funaki and Yamato(1999)). For any $\mathcal{P} = \{S_1, S_2, ..., S_k\}$, there exists a unique equilibrium $(x_{S_1}^*, x_{S_2}^*, ..., x_{S_n}^*)$ under $\mathcal{P}$ which satisfies

\[
f'(\sum_{j=1}^{k} x_{S_j}^*) + \frac{(k-1)f(\sum_{j=1}^{k} x_{S_j}^*)}{\sum_{j=1}^{k} x_{S_j}^*} = kq, \quad x_{S_i}^* = \frac{\sum_{j=1}^{k} x_{S_j}^*}{k} > 0 \forall i.
\]

Given a coalition structure $\mathcal{P} = \{S_1, ..., S_k\}$, let $(x_{S_1}^*(\mathcal{P}), ..., x_{S_k}^*(\mathcal{P}))$ be a unique equilibrium under $\mathcal{P}$ and let $x_N^*(\mathcal{P}) = \sum_{i=1}^{k} x_{S_i}^*(\mathcal{P})$. Moreover, let $m_{S_i}^*(\mathcal{P}) = m_{S_i}(x_{S_1}^*(\mathcal{P}), ..., x_{S_k}^*(\mathcal{P}))$ be the equilibrium income of coalition $S_i$ for $i = 1, ..., k$ and therefore $m_N^*(\mathcal{P}) = \sum_{i=1}^{k} m_{S_i}(x_{S_1}^*(\mathcal{P}), ..., x_{S_k}^*(\mathcal{P}))$.

Proposition 5 (Funaki and Yamato(1999)). For two coalition structures $\mathcal{P}_k = \{S_1, S_2, ..., S_k\}$ and $\mathcal{P}_{k'} = \{S'_1, S'_2, ..., S'_{k'}\}$ with $k < k'$,

\[
x_N^*(\mathcal{P}_k) < x_N^*(\mathcal{P}_{k'}), \quad \frac{m_N^*(\mathcal{P}_k)}{n} > \frac{m_N^*(\mathcal{P}_{k'})}{n},
\]

$S \in \mathcal{P}_k$ and $S \in \mathcal{P}_{k'} \implies m_S^*(\mathcal{P}_k) > m_S^*(\mathcal{P}_{k'})$.

Proposition 5 says that as the number of coalitions decreases, the total amount of labor input decreases, whereas the average income increases. Also, if the number of coalitions in one coalition structure is smaller than that in another coalition structure and coalition $S$ belongs to both coalition structures, then the income of coalition $S$ under the former structure is larger than that under the latter.

We assume that for a common pool resource game, the feasible payoff vector is given by $u_i(\mathcal{P}) = \frac{m_{S_i}^*(\mathcal{P})}{|S_j|}$ $\forall i \in S_j, \forall S_j \in \mathcal{P}$. It is natural to consider this because of the symmetry of players.

The following result will be useful.

Lemma 1. In a common pool resource game, let a coalition structure $\mathcal{P} \neq \mathcal{P}^I$ be given. Without loss of generality, denote the coalition structure by $\mathcal{P} = \{S_1, S_2, ..., S_k\}$, where $S_1 = \{1, 2, ..., r\}$, $2 \leq r \leq n$, and $1 \leq k \leq n - r + 1$.

Suppose that the coalition $S_1$ is divided into two subcoalitions, $S'_1 \equiv \{1, ..., \ell\}$ and $S''_1 \equiv \{\ell + 1, ..., r\}$, where $1 \leq \ell \leq r/2$. All other players do not change their behavior in coalition formation. Denote this coalition structure $\mathcal{P}' = \{S'_1, S''_1, S_2, ..., S_k\}$. Then $u_1(\mathcal{P}') > u_1(\mathcal{P})$ if $k^2/(k + 1)^2 \geq \ell/r$, in particular, if (i) $r = n$ and $n/\ell \geq 4$, (ii) $3 \leq r \leq n - 1$ and $\ell/r \leq 4/9$, or (iii) $r = 2$ and $k \geq 3$.

Proof. By Proposition 4,

\[
u_1(\mathcal{P}) = m_{S_1}^*(\mathcal{P})/r = \frac{f(x_N^*(\mathcal{P})) - qx_N^*(\mathcal{P})}{rk} = \frac{f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})}{rk^2},
\]

\[
u_1(\mathcal{P}') = m_{S_1}^*(\mathcal{P}')/\ell = \frac{f(x_N^*(\mathcal{P}')) - qx_N^*(\mathcal{P}')}{\ell(k+1)} = \frac{f(x_N^*(\mathcal{P}')) - f'(x_N^*(\mathcal{P}'))x_N^*(\mathcal{P})}{\ell(k+1)^2}.
\]
Therefore,

\[ u_1(P') - u_1(P) = \frac{rk^2\{f(x_n^*(P')) - f'(x_n^*(P'))x_n^*(P')\} - \ell(k+1)^2\{f(x_n^*(P)) - f'(x_n^*(P))x_n^*(P)\}}{rk^2(k+1)^2}. \]

Here, \(0 < f(x_n^*(P)) - x_n^*(P)f'(x_n^*(P')) < f(x_n^*(P')) - x_n^*(P')f'(x_n^*(P'))\) holds because \(f(x) - xf'(x)\) is increasing for \(x > 0\), and \(x_n^*(P) < x_n^*(P')\) by Proposition 5. Therefore, \(u_1(P') > u_1(P)\) if \(A \equiv rk^2 - \ell(k+1)^2 \geq 0\). This condition is satisfied in the following cases.

**Case 1.** \(r = n\) and \(n/\ell \geq 4\) Note that \(r = n\) if and only if \(k = 1\). Hence, \(A = n - 4\ell \geq 0\) if \(n/\ell \geq 4\).

**Case 2.** \(3 \leq r \leq n - 1\) and \(4/9 \geq \ell/r\): Since \(r \neq n, k \geq 2\). Also, \(k^2/(k+1)^2\) is increasing for \(k > 0\). Therefore, \(k^2/(k+1)^2 \geq 4/9\). Accordingly, if \(4/9 \geq \ell/r\), then \(A \geq 0\).

**Case 3.** \(r = 2\) and \(k \geq 3\): Since \(r = 2, \ell = 1\). Thus \(A = (k - 1)^2 - 2 \geq 2 > 0\).

Q.E.D.

### 4.2 Credible Cores and EBA Cores in Common Pool Resource Games

We now examine the credible and EBA cores of a common pool resource game. First of all, the credible core and the EBA core could be different as the following example illustrates:

**Example 5.** Suppose the production function \(f(x)\) is given by \(f(x) = \sqrt{x}\).

(1) Let \(n = 4\). The credible and EBA cores are the same. They consist of all coalition structures with even numbers of coalitions.

(2) Let \(n = 5\). The credible core consist of all coalition structures with odd numbers of coalitions. On the other hand, neither the grand coalition structure \(P^N\) nor the coalition structures \{1; 1; 3\} are EBA’s, although they contain odd numbers of coalitions. The EBA core consists of the singleton coalition structure \(P^I\), the coalition structures \{1; 2; 2\}, and the coalition structures consisting of \((n - 1)\)-person coalition and one-person coalition \(P^{N\setminus i} = \{1; 4\}\).

(3) Let \(n = 6\). The credible core consists of all coalition structures containing even numbers of coalitions. On the other hand, the coalition structures \{1; 1; 1; 3\}, \{2; 4\}, nor \(P^{N\setminus i} = \{1; 5\}\) are not an EBA, although they contain even numbers of coalitions. The EBA core consists of the singleton coalition structure \(P^I\), the coalition structures \{3; 3\}, \{1; 1; 2; 2\}, and the grand coalition structure \(P^N\).

Concerning the credible core, we can derive the following general property:

**Theorem 1.** Let \(n \geq 4\).

a) Suppose \(n\) is odd. A coalition structure \(P\) is credible if and only if it contains an odd number of coalitions.

b) Suppose \(n\) is even. A coalition structure \(P\) is credible if and only if it contains an even number of coalitions.
Proof. Consider the case $n \geq 5$ first. Proposition 5 implies that for the payoff vector $z$ in $\mathcal{F}(\mathcal{P})$ with $\mathcal{P} \neq \mathcal{P}'$, $\mathcal{P}$ is directly dominated by some $T$ under some $\mathcal{P}'$. Consider any coalition structure $\mathcal{P}$ such that $|\mathcal{P}| - 1 = |\mathcal{P}'|$ and $\mathcal{P}'$ is finer than $\mathcal{P}$. Since $\mathcal{P}'$ is credible by definition, the above result implies that $\mathcal{P}$ is directly dominated by the finer credible coalition structure $\mathcal{P}'$. This means that $\mathcal{P}$ is not credible. The set of such $\mathcal{P}$ is denoted by $\mathcal{P}^2$. That is,

$$\mathcal{P}^2 = \{ \mathcal{P} ||\mathcal{P}| - 1 = |\mathcal{P}'| \text{ and } \mathcal{P}' \text{ is finer than } \mathcal{P} \}. $$

By a simple consideration, we have $\mathcal{P}^2 = \{ \mathcal{P} ||\mathcal{P}| = n - 1 \}$. The above result directly implies that any $\mathcal{P}' \in \mathcal{P}^3$ is credible because any $\mathcal{P} \in \mathcal{P}^2$ is not credible, where

$$\mathcal{P}^3 = \{ \mathcal{P}' ||\mathcal{P}'| - 1 = |\mathcal{P}| \text{ for some } \mathcal{P} \in \mathcal{P}^2 \text{ and } \mathcal{P} \text{ is finer than } \mathcal{P}' \}$$

$$= \{ \mathcal{P}' | |\mathcal{P}'| = n - 2 \}.$$

This consideration implies that any $\mathcal{P} \in \mathcal{P}^m$ is credible if $m = n - 2k(k = 0, 1, 2, ...)$, and not credible if $m = n - 2k - 1(k = 0, 1, 2, ...)$, Since $m = n - 2k$ is odd if $n$ is odd, $\mathcal{P}$ consisting of odd number of coalitions is credible. Since $m = n - 2k$ is even if $n$ is even, $\mathcal{P}$ consisting of even number of coalitions is credible.

For the case $n = 4$, put $r = 2$ and $k = 3$ in Lemma 1. This implies $\mathcal{P} \in \mathcal{P}^2$ is not credible because $\mathcal{P}'$ is credible. Then $\mathcal{P} \in \mathcal{P}^3$ is credible. Put $r = 3$ and $\ell = 1$ in Lemma 1. This implies $\mathcal{P} \in \mathcal{P}^4$ is not credible because $\mathcal{P} \in \mathcal{P}^3$ is credible.

Q.E.D.

Theorem 1 means that if the number of players is odd, then the credible core consists of all coalition structures with odd numbers of coalitions. In particular, the grand coalition structure is credible. On the other hand, if the number of players is even, then the credible core consists of all coalition structures with even numbers of coalitions. In this case, however, the grand coalition structure is not credible. Regarding the EBA core, it seems to be hard to find such a general property as Example 5 illustrates.

In both the credible and EBA cores, only breaking up is allowed for coalitions. In the next subsection, we consider sequential stability in cases in which coalitions can both break up and merge. In such situations, we will see that the grand coalition structure is more likely to be stable.

4.3 Sequentially Stable Coalition Structures in Common Pool Resource Games

We turn to apply our concept of sequentially stability to a common pool resource game. We will examine sequential stability of the grand coalition structure. The

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1. Even when the number of players is fixed, the grand coalition structure may or may not be an EBA, depending on the shape of the production function. For example, let $n = 8$ and $f(x) = x^\alpha$. The grand coalition structure $\mathcal{P}^N$ is an EBA when $\alpha = 0.2, 0.5, 0.8$, but not when $\alpha = 0.001, 0.9, 0.995$. 

---
following lemma will be useful below, which gives a sufficient condition for which all players prefer the grand coalition structure to another coalition structure.

**Lemma 2.** In a common pool resource game, let a coalition structure \( \mathcal{P} \) be given. Without loss of generality, denote the coalition structure by \( \mathcal{P} = \{ S_1, S_2, S_3, ..., S_k \} \), where \( |S_1| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq ... \leq |S_k| = r_k \). Let

\[
B(k) \equiv \left\{ f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P}) \right\}/[k^2 \{ f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P})x_N^*(\mathcal{P}^N)) \}],
\]

where \( \mathcal{P}^N = \{ 1, 2, ..., n \} \) is the grand coalition structure. Then for each \( i \in N \),

\[
u_i(\mathcal{P}) \leq \nu_i(\mathcal{P}^N) \text{ if } B(k) \leq r_1/n.
\]

**Proof.** By Proposition 4,

\[
u_i(\mathcal{P}) = m_{S_i}(\mathcal{P})/r_j = \frac{[f(x_N^*(\mathcal{P})) - qx_N^*(\mathcal{P})]}{(r_j k^2)},
\]

for \( i \in S_j \) and \( j = 1, ..., k \). Notice that for the grand coalition structure \( \mathcal{P}^N \), \( k = 1 \) and \( r_1 = n \), so that \( u_i(\mathcal{P}^N) = \frac{[f(x_N^*(\mathcal{P}^N)) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P}^N)]]}{n} \) for \( i \in N \). We also remark that a player belonging to the smallest coalition, \( S_1 \), obtains the highest payoff among all players, that is, the payoff of each player \( i \), \( u_i(\mathcal{P}) \), is less than or equal to \( u_j(\mathcal{P}) = m_{S_1}(\mathcal{P})/r_1 \) for \( j \in S_1 \). Therefore, each \( i \in N \), \( u_i(\mathcal{P}) \leq u_i(\mathcal{P}^N) \) if \( B(k) = \{ f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P}) \}/[k^2 \{ f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P}^N) \}] \leq r_1/n \).

Q.E.D.

We will identify a condition for which the grand coalition structure is sequentially stable. We will give two theorems. The basic idea behind the proofs of the theorems are the same. The proof consists of 4 steps. The outline of the proof is as follows:

1. The grand coalition structure \( \mathcal{P}^N \) sequentially dominate some key coalition structure \( \mathcal{P}^* \).
2. Every coalition structure \( \mathcal{P} \) such that \( |\mathcal{P}| = |\mathcal{P}^*| \) is sequentially dominated by \( \mathcal{P}^N \).
3. Every coalition structure \( \mathcal{P} \) such that \( |\mathcal{P}| < |\mathcal{P}^*| \) other than \( \mathcal{P}^N \) is sequentially dominated by \( \mathcal{P}^N \).
4. Every coalition structure \( \mathcal{P} \) such that \( |\mathcal{P}| > |\mathcal{P}^*| \) is sequentially dominated by \( \mathcal{P}^N \).

First consider a case \( n = 2^m \ (m \geq 2) \). We say \( \mathcal{P} \) is a \( k \)-th stage coalition structure if \( |\mathcal{P}| = k \).

**Theorem 2.** Let \( n = 2^m \ (m \geq 2) \). If \( B(k) < 1/2^{k-1} \) for all \( k(2, ..., m, m + 1) \), the grand coalition structure is sequentially stable.

**Proof.** We have to show that every coalition structure other than the grand coalition structure \( \mathcal{P}^N \) is sequentially dominated by \( \mathcal{P}^N \). In the following, we denote a coalition structure \( \mathcal{P} = \{ S_1, S_2, S_3, ..., S_k \} \), where \( |S_1| = r_1 \leq |S_2| = r_2 \leq ... \leq |S_k| = r_k \).
Consider a coalition structure $\mathcal{P}^*$ consisting of the following $(m + 1)$ coalitions: two 1-person coalitions, one 2-person coalition, one 4-person coalition, one 8-person coalition, ..., and one $2^{m-1}$-person coalition. This coalition structure is denoted by \{1; 1; 2; 4; 8; ...; $2^{m-1}$\}.

The proof consists of four steps.

**Step 1** $\mathcal{P}^*$ is sequentially dominated by $\mathcal{P}^N$.

Consider a sequence of coalition structures $\{\mathcal{P}_t\}_{t=0}^m$ such that $\mathcal{P}_0 = \mathcal{P}^*$, $\mathcal{P}_m = \mathcal{P}^N$, and the two coalitions of the smallest size in $\mathcal{P}_t$ merge into one coalition in $\mathcal{P}_{t+1}$ for $t = 0, 1, 2, ..., m - 1$. This sequence is expressed by

$$\mathcal{P}_0 = \mathcal{P}^* = \{1; 1; 2; 4; 8; ...; 2^{m-2}; 2^{m-1}\} \rightarrow \mathcal{P}_1 = \{2; 2; 4; 8; ...; 2^{m-2}; 2^{m-1}\}$$

$$\rightarrow \mathcal{P}_2 = \{4; 4; 8; ...; 2^{m-2}; 2^{m-1}\} \rightarrow ... \rightarrow \mathcal{P}_{m-2} = \{2^{m-2}; 2^{m-2}; 2^{m-1}\} \rightarrow \mathcal{P}_{m-1} = \{2^{m-1}; 2^{m-1}\} \rightarrow \mathcal{P}_m = \mathcal{P}^N = \{2^m\}$$

First, it follows from Lemma 2 that the 2nd stage coalition structure $\mathcal{P}_{m-1} = \{2^{m-1}; 2^{m-1}\}$ is dominated by $\mathcal{P}^N$, since $r_1/n = 2^{m-1}/2^m = 1/2 > B(2)$ by the hypothesis.

Next, it follows from Lemma 2 that the 3rd stage coalition structure $\mathcal{P}_{m-2} = \{2^{m-2}; 2^{m-2}; 2^{m-1}\}$ is sequentially dominated by $\mathcal{P}^N$, since $r_1/n = 2^{m-2}/2^m = 1/4 > B(3)$ by the hypothesis.

In general, for $k = 2, ..., m, m + 1$, it follows from Lemma 2 that the $k$-th stage coalition structure $\mathcal{P}_{m-k+1} = \{2^{m-k+1}; 2^{m-k+1}; 2^{m-k+2}; 2^{m-k+2}; 2^{m-k+3}; ...; 2^{m-1}\}$ is sequentially dominated by $\mathcal{P}^N$, since $r_1/n = 2^{m-k+1}/2^m = 1/2^{k-1} > B(k)$ by the hypothesis.

Therefore, the $(m + 1)$-th stage coalition structure $\mathcal{P}_0 = \mathcal{P}^* = \{1; 1; 2; 4; ...; 2^{m-1}\}$ is sequentially dominated by $\mathcal{P}^N$.

**Step 2** Every $(m + 1)$-th stage coalition structure is sequentially dominated by $\mathcal{P}^N$.

Take any $(m + 1)$-stage coalition structure $\mathcal{P}$.

First we consider a sequence $\{\mathcal{P}_t\}_{t=0}^T$ such that

1) $\mathcal{P}_0 = \mathcal{P} = \{r_1; r_2; r_3; ...; r_{m-1}; r_m; r_{m+1}\}$

2) $\mathcal{P}_T = \{1; 1; 1; ...; 1; 2^m - m\}$, where $|\mathcal{P}_T| = m + 1$.

3) If $t$ is zero or even, then the largest and the second largest coalitions in $\mathcal{P}_t$ merge into one coalition in $\mathcal{P}_{t+1}$.

4) If $t$ is odd, then one person belonging to the largest coalition in $\mathcal{P}_t$ deviates and forms one person coalition in $\mathcal{P}_{t+1}$.

Then the sequence $\{\mathcal{P}_t\}_{t=0}^T$ of coalition structures is given by:

$$\mathcal{P}_0 = \{r_1; r_2; r_3; ...; r_{m-1}; r_m; r_{m+1}\} \quad ((m + 1)$-th stage)$$

$$\rightarrow \mathcal{P}_1 = \{r_1; r_2; r_3; ...; r_{m-1}; r_m + r_{m+1}\} \quad (m$-th stage)$$

$$\rightarrow \mathcal{P}_2 = \{1; r_1; r_2; r_3; ...; r_{m-1}; r_m + r_{m+1} - 1\} \quad ((m + 1)$-th stage)$$

$$\rightarrow ... \rightarrow ...$$
\[ \rightarrow \mathcal{P}_{T-2} = \{1; 1; 1; \ldots; 1; r_1; \sum_{k=2}^{m+1} r_k - m + 1\} \text{ } ((m+1)\text{-th stage}) \]

\[ \rightarrow \mathcal{P}_{T-1} = \{1; 1; 1; \ldots; 1; \sum_{k=1}^{m+1} r_k - m + 1\} \text{ } (m\text{-th stage}) \]

\[ \rightarrow \mathcal{P}_T = \{1; 1; 1; \ldots; 1; \sum_{k=1}^{m+1} r_k - m\} = \{1; 1; 1; \ldots; 1; 2^m - m\} \text{ } ((m+1)\text{-th stage}) \]

Next consider \( \{\mathcal{P}_t\}_{t=T}^{T'} \) such that

1) \( \mathcal{P}_T = \{1; 1; 1; \ldots; 1; 2^m - m\}, \)

2) \( \mathcal{P}_{T+T'} = \mathcal{P}^* = \{1; 1; 2; 4; 8; \ldots; 2^{m-2}; 2^{m-1}\}, \)

3) If \( t = T + \lambda \) and \( \lambda \) is zero or even \((\lambda \leq T' - 2), \) then the smallest coalition of more than one members and a 1-person coalition in \( \mathcal{P}_{T+\lambda} \) merge into one coalition in \( \mathcal{P}_{T+\lambda+1}. \)

4) If \( t = T + \lambda \) and \( \lambda \) is odd \((\lambda \leq T' - 2), \) then \( 2^{m-\frac{\lambda+1}{2}} \) persons in the coalition of \( 2^{m-\frac{\lambda+1}{2}+1} - (m - \frac{\lambda+1}{2}) \) persons in \( \mathcal{P}_{T+\lambda} \) deviate and form a coalition in \( \mathcal{P}_{T+\lambda+1}. \) Note that \( 2^{m-\frac{\lambda+1}{2}+1} - (m - \frac{\lambda+1}{2}) \geq 1. \)

5) If \( t = T + T' - 1, \) then two one-person coalitions in \( \mathcal{P}_{T+T'-1} \) merge into one coalition in \( \mathcal{P}_{T+T'}. \)

This sequence \( \{\mathcal{P}_t\}_{t=T}^{T'} \) of coalition structures is given by:

\[ \mathcal{P}_T = \{1; 1; 1; \ldots; 1; 1; 1; 2^m - m\} \text{ } ((m+1)\text{-th stage}) \]

\[ \rightarrow \mathcal{P}_{T+1} = \{1; 1; 1; \ldots; 1; 1; 1; 2^m - m + 1\} \text{ } (m\text{-th stage}) \]

\[ \rightarrow \mathcal{P}_{T+2} = \{1; 1; 1; \ldots; 1; 1; 1; 2^m - m + 1 - 2^{m-1}; 2^{m-1}\} \]

\[ = \{1; 1; 1; \ldots; 1; 1; 1; 2^{m-1} - m + 1; 2^{m-1}\} \text{ } ((m+1)\text{-th stage}) \]

\[ \rightarrow \mathcal{P}_{T+3} = \{1; 1; 1; \ldots; 1; 1; 2^{m-1} - m + 2; 2^{m-1}\} \text{ } (m\text{-th stage}) \]

\[ \rightarrow \mathcal{P}_{T+4} = \{1; 1; 1; \ldots; 1; 1; 2^{m-1} - m + 2 - 2^{m-2}; 2^{m-2}; 2^{m-1}\} \]

\[ = \{1; 1; 1; \ldots; 1; 1; 2^{m-2} - m + 2; 2^{m-2}; 2^{m-1}\} \text{ } ((m+1)\text{-th stage}) \]

\[ \rightarrow \mathcal{P}_{T+5} = \{1; 1; 1; \ldots; 1; 2^{m-2} - m + 3; 2^{m-2}; 2^{m-1}\} \text{ } (m\text{-th stage}) \]

\[ \rightarrow \ldots \rightarrow \ldots \]

\[ \rightarrow \mathcal{P}_{T+T'-1} = \{1; 1; 1; 4; 8; \ldots; 2^{m-3}; 2^{m-2}; 2^{m-1}\} \text{ } ((m+1)\text{-th stage}) \]

\[ \rightarrow \mathcal{P}_{T+T'} = \{1; 1; 2; 4; 8; \ldots; 2^{m-3}; 2^{m-2}; 2^{m-1}\} \text{ } (m\text{-th stage}) \]

This sequence ends at the coalition structure \( \mathcal{P}_T = \mathcal{P}^*. \)
Hence if we combine two sequences \( \{P_t\}_{t=0}^{T} \) and \( \{P_t\}_{t=T}^{T+T'} \), we can get a sequence \( \{P_t\}_{t=0}^{T+T'} \) from any \((m+1)\)-th stage coalition structure \( P \) to \( P^* \). Note that only \((m+1)\)-th stage and \(m\)-th stage coalition structures appear in this sequence.

Each member of any coalition in \((m+1)\)-th stage coalition structure prefers the payoff under the grand coalition structure \( P^N \) to the payoff under the \((m+1)\)-th stage coalition structure because of \( B(m+1) < 1/2^m \). Moreover any deviating coalition in the process from \(m\)-th stage coalition structure to \((m+1)\)-th stage coalition structure consists of at least two players. Each member of such a deviating coalition prefers the payoff in the grand coalition structure \( P^N \) to the payoff in the \(m\)-th stage coalition structure, because of \( B(m) < 2/2^m = 1/2^{m-1} \) by Lemma 2.

Therefore if we combine this sequence \( \{P_t\}_{t=0}^{T} \) and a sequence from \( P_{T+T'} = P^* \) to \( P^N \), every coalition structure in the sequence \( \{P_t\}_{t=0}^{T+T'} \) is sequentially dominated by \( P^N \). And so is the \((m+1)\)-th stage coalition structure \( P \). This completes the proof of Step 2.

**Step 3** Every coalition structure \( P \) of less than \( m+1 \) coalitions other than the grand coalition structure \( P^N \) is sequentially dominated by \( P^N \).

First, we show that each member of a coalition of the maximal size in any coalition structure \( P \) prefers her payoff under \( P^N \) to her payoff under \( P \). Denote \( P \) by \( P = \{S_1, S_2, S_3, \ldots, S_k\} \), where \(|S_1| = r_1 \leq |S_2| = r_2 \leq |S_3| = r_3 \leq \ldots \leq |S_k| = r_k \). Because \( r_k \geq r_i \) for all \( r_i \), \( kr_k \geq \sum_{i=1}^{k} r_i = n \), that is, \( r_k/n \geq 1/k \). Since \( B(k) < 1/2^{k-1} \), it follows that \( r_k/n \geq 1/k \geq 1/2^{k-1} > B(k) \). By Lemma 2, we have the desired result.

Take any coalition structure \( P \) of less than \( m+1 \) coalitions other than \( P^N \). Consider the following sequence \( \{P_t\} \) starting from \( P \) to some \((m+1)\)-stage coalition structure \( P' \): one person in a coalition of the maximal size in \( P \) deviates and forms a 1-person coalition in \( P_{t+1} \). Notice that such a person in \( P \) prefers her payoff under \( P^N \) to her payoff under \( P_t \), as shown above. Moreover, it is easy to construct a sequence of coalition structures from \( P \) to \( P^N \) by combining the above sequence from \( P \) to \( P' \) and the sequence from \( P' \) to \( P^N \) in Step 2. These imply that \( P \) is sequentially dominated by \( P^N \).

**Step 4** Every coalition structure \( P \) of more than \( m+1 \) coalitions is sequentially dominated by \( P^N \).

Take any \( k \)-th stage coalition structure \( P \) of more than \( m+1 \) coalitions. Since \( B(k) \) is a decreasing function, \( B(k) < B(m+1) < 1/2^m = 1/n \leq r_i/n \) holds for any \( r_i \geq 1 \). This together with Lemma 1 imply that each member of any coalition in \( P \) prefers her payoff under the grand coalition structure \( P^N \) to her payoff under \( P \).

Consider a sequence \( \{P_t\} \) starting from \( P \) to some \((m+1)\)-stage coalition structure \( P' \) such that two coalitions in \( P_t \) merge into one coalition in \( P_{t+1} \). Notice that each member in these two coalitions in \( P_t \) prefers her payoff under \( P^N \) to her payoff under \( P_t \), as shown above. Moreover, it is easy to construct a sequence of coalition structures from \( P \) to \( P^N \) by combining the above sequence from \( P \) to \( P' \) and the sequence from \( P' \) to \( P^N \) in Step 2. These imply that \( P \) is sequentially dominated by \( P^N \).

Q.E.D.

Next consider a case that \( n = 2^m + l \) \((m \geq 2, 0 \leq l \leq 2^m - 1) \). This theorem is an extension of Theorem 2.
Theorem 3. Let \( n = 2^m + l \) \((m \geq 2, 0 \leq l \leq 2^m - 1)\). If the inequalities
\[
B(2^{m-h-1} + 2) < \frac{2^{h-1}}{n} \quad (h = 1, 2, \ldots, m-1)
\]
and \( B(2) < \frac{2^{m-1}}{n} \) hold, and \( B(k) \) is monotonically decreasing in \( k \), then the grand coalition structure is sequentially stable.

Proof. Consider a coalition structure \( \mathcal{P}^{**} = \{1; 1; 2; 2; 2; \ldots; 2; 2; 2^{m-1} + l\} \) consisting of \( 2^{m-2} + 2 \) coalitions instead of \( \mathcal{P}^* = \{1; 1; 2; 4; 8; \ldots; 2^{m-1}\} \) in Theorem 2.

(Step 1) \( \mathcal{P}^{**} \) is sequentially dominated by \( \mathcal{P}^N \).

We have to find a sequence of coalition structures \( \{\mathcal{P}_t\}_{t=1}^{2^{m-2}+2} \) from \( \mathcal{P}_1 = \mathcal{P}^{**} \) to \( \mathcal{P}_{2^{m-2}+2} = \mathcal{P}^N \). We will show the following is a domination sequence of coalition structures.

\[
\begin{align*}
\mathcal{P}^{**} &= \mathcal{P}_1 = \{1; 1; 2; 2; 2; 2; \ldots; 2; 2; 2^{m-1} + l\} \quad (2^{m-2} + 2)-th stage \\
&\rightarrow \mathcal{P}_2 = \{2; 2; 2; 2; 2; 2; 2; 2; 2^{m-1} + l\} \quad (2^{m-2} + 1)-th stage \\
&\rightarrow \mathcal{P}_3 = \{4; 2; 2; 2; 2; \ldots; 2; 2; 2^{m-1} + l\} \quad (2^{m-3})-th stage \\
&\rightarrow \mathcal{P}_4 = \{4; 4; 2; \ldots; 2; 2; 2^{m-1} + l\} \quad (2^{m-3} - 1)-th stage \\
&\rightarrow \ldots \rightarrow \ldots \\
&\rightarrow \mathcal{P}_{2^{m-3}+1} = \{4; 4; 4; 4; \ldots; 2; 2; 2^{m-1} + l\} \quad (2^{m-3} + 1)-th stage \\
&\rightarrow \mathcal{P}_{2^{m-3}+2} = \{4; 4; 4; 4; \ldots; 4; 2^{m-1} + l\} \quad (2^{m-3} + 2)-th stage \\
&\rightarrow \mathcal{P}_{2^{m-3}+3} = \{8; 4; 4; \ldots; 4; 2^{m-1} + l\} \quad (2^{m-3} - 1)-th stage \\
&\rightarrow \ldots \rightarrow \ldots \\
&\rightarrow \mathcal{P}_{2^{m-3}+2^{m-4}+1} = \{8; 8; \ldots; 8; 4; 4; 2^{m-1} + l\} \quad (2^{m-4} + 1)-th stage \\
&\rightarrow \mathcal{P}_{2^{m-3}+2^{m-4}+2} = \{8; 8; \ldots; 8; 2^{m-1} + l\} \quad (2^{m-4} + 2)-th stage \\
&\rightarrow \ldots \rightarrow \ldots \\
&\rightarrow \mathcal{P}_{2^{m-3}+2^{m-4}+2^{m-3}+1} = \{2^h; 2^h; 2^h; 2^h; \ldots; 2^h; 2^{h-1}; 2^{h-1}; 2^{m-1} + l\} \quad (2^{m-h-1} + 2)-th stage \\
&\rightarrow \mathcal{P}_{2^{m-3}+2^{m-4}+2^{m-3}+2^{m-h-1}+2} = \{2^h; 2^h; 2^h; 2^h; \ldots; 2^h; 2^h; 2^{m-1} + l\} \quad (2^{m-h-1} + 1)-th stage \\
&\rightarrow \ldots \rightarrow \ldots \\
&\rightarrow \mathcal{P}_{2^{m-2}+2} = \{2^{m-3}; 2^{m-3}; 2^{m-3}; 2^{m-3}; 2^{m-1} + l\} \quad (5-th stage) \\
&\rightarrow \mathcal{P}_{2^{m-2}+1} = \{2^{m-3}; 2^{m-3}; 2^{m-3}; 2^{m-1} + l\} \quad (4-th stage) \\
&\rightarrow \mathcal{P}_{2^{m-2}} = \{2^{m-2}; 2^{m-2}; 2^{m-1} + l\} \quad (3-th stage)
\end{align*}
\]
For $t = 1$, the payoff of every player in the two singletons under $\mathcal{P}_1$ is smaller than that under the final coalition structure $\mathcal{P}^N$ by Lemma 1 because $\frac{1}{n} > B(2^{m-2} + 2)$, which is given by ($\ast$) for $h = 1$. Remark that $|\mathcal{P}_1| = 2^{m-2} + 2$.

For $t = 2, 3, \ldots, 2^{m-3} + 1$, the payoff of every player in 2-person coalitions under $\mathcal{P}_t$, $(t = 3, \ldots, 2^{m-2})$, which is the minimal size among the coalitions, is smaller than that under the final coalition structure $\mathcal{P}^N$ by Lemma 1 because $\frac{2}{n} > B(2^{m-3} + 2) > B(2^{m-3} + 3) > \ldots > B(2^{m-2} + 1)$. Here this inequality is obtained by $\frac{2}{n} > B(2^{m-3} + 2)$ and the monotonicity of $B(k)$.

Let $h \in \{3, 4, \ldots, m - 2\}$. For $t = (2^{m-3} + 2^{m-4} + \ldots + 2^{m-h}) + 2, 2^{m-3} + 2^{m-4} + \ldots + 2^{m-h} + 3, \ldots, (2^{m-3} + 2^{m-4} + \ldots + 2^{m-h}) + 1$, the payoff of every member in 2-person coalitions under $\mathcal{P}_t$, $t = (2^{m-3} + 2^{m-4} + \ldots + 2^{m-h}) + 2, 2^{m-3} + 2^{m-4} + \ldots + 2^{m-h} + 3, \ldots, (2^{m-3} + 2^{m-4} + \ldots + 2^{m-h}) + 1$ is smaller than that under $\mathcal{P}^N$ by Lemma 1, because $\frac{2^{m-1}}{n} > B(2^{m-h-1} + 2)$ and $B(k)$ is decreasing.

For $t = 2^{m-2} - 1$ or $2^{m-2} - 2$, which correspond to the case of $h = m - 2$ above, the payoff of every member in two 2^{m-2}-person coalitions under $\mathcal{P}^{2^{m-2}-1}$ is smaller than that under $\mathcal{P}^N$ because $\frac{2^{m-3}}{n} > B(4) = B(2^1 + 2) = B(5)$.

For $t = 2^{m-2}$, the payoff of every member in two 2^{m-2}-person coalitions under $\mathcal{P}_{2^{m-2}}$ is smaller than that under $\mathcal{P}^N$ because $\frac{2^{m-2}}{n} > B(3) = B(2^0 + 2)$.

For $t = 2^{m-2} + 1$, the payoff of every member in the two coalitions under $\mathcal{P}_{2^{m-2}+1}$ is smaller than that under $\mathcal{P}^N$ because $\frac{2^{m-2}}{n} > B(2)$.

Thus we get a sequence from $\mathcal{P}^* \to \mathcal{P}^N$.

**Step 2** Every $(2^{m-2} + 2)$-th stage coalition structure is sequentially dominated.

We denote $M = 2^{m-2} + 2$. Take any $M$-stage coalition structure $\mathcal{P}$.

First we consider a sequence $\{\mathcal{P}_t\}_{t=0}^{T}$ such that

1. $\mathcal{P}_0 = \mathcal{P} = \{r_1; r_2; r_3; \ldots; r_{M-2}; r_{M-1}; r_M\}$, $r_1 \leq r_2 \leq r_3 \leq \ldots \leq r_{M-1} \leq r_M$.
2. $\mathcal{P}_T \in \{1; 1; 1; \ldots; 1; n - M + 1\}$, where $|\mathcal{P}_T| = M$.
3. If $t$ is zero or even, then one person belonging to the largest coalition in $\mathcal{P}_t$ deviates and forms one person coalition in $\mathcal{P}_{t+1}$.
4. If $t$ is odd, then the largest and the second largest coalitions in $\mathcal{P}_t$ merge into one coalition in $\mathcal{P}_{t+1}$.

Then the sequence $\{\mathcal{P}_t\}_{t=0}^{T}$ of coalition structures is given by:

$\mathcal{P}_0 = \{r_1; r_2; r_3; \ldots; r_{M-2}; r_{M-1}; r_M\}$ $\text{(}M\text{-th stage)}$

$\Rightarrow \mathcal{P}_1 = \{1; r_1; r_2; r_3; \ldots; r_{M-2}; r_{M-1}; r_M\}$ $\text{(}M + 1\text{-th stage)}$

$\Rightarrow \mathcal{P}_2 = \{1; r_1; r_2; r_3; \ldots; r_{M-2}; r_{M-1} + 1; r_M - 1\}$ $\text{(}M\text{-th stage)}$

$\Rightarrow \mathcal{P}_3 = \{1; 1; r_1; r_2; r_3; \ldots; r_{M-2}; r_{M-1} + r_M - 2\}$ $\text{(}(M + 1)\text{-th stage)}$

$\Rightarrow \ldots \Rightarrow \ldots$

$\Rightarrow \mathcal{P}_{T-2} = \{1; 1; 1; \ldots; 1; r_1; \sum_{k=2}^{M} r_k - M + 2\}$ $\text{(}M\text{-th stage)}$
Next consider \( \{P_t\}_{t=T}^{T+T'} \) such that
1) \( P_T = \{1; 1; 1; \ldots; 1; n - M + 1\} \) (M-th stage)
2) \( P_{T+1} = \{1; 1; 1; 1; 1; 1; n - M\} \) ((M + 1)-th stage)
3) \( P_{T+2} = \{1; 1; 1; \ldots; 1; 1; 1; n - M\} \) (M-th stage)
4) \( P_{T+3} = \{1; 1; 1; 1; 1; 2; n - M - 1\} \) ((M + 1)-th stage)
5) \( P_{T+4} = \{1; 1; 1; 1; 1; 2; n - M - 1\} \) (M-th stage)
6) \( P_{T+5} = \{1; 1; 1; 1; 1; 2; 2; n - M - 2\} \) ((M + 1)-th stage)

If \( t = T + \lambda \) and \( \lambda \) is 0 or even, then 1-person in the largest coalition in \( P_{T+\lambda} \) deviate and form a singleton in \( P_{T+\lambda+1} \).
4) If \( t = T + \lambda \) and \( \lambda \) is odd, then two 1-person coalitions in \( P_{T+\lambda} \) merge into one coalition in \( P_{T+\lambda+1} \).

This sequence \( \{P_t\}_{t=T}^{T+T'} \) of coalition structures is given by:

\[
\begin{align*}
P_T &= \{1; 1; 1; \ldots; 1; 1; 1; n - M + 1\} \quad (M\text{-th stage}) \\
P_{T+1} &= \{1; 1; 1; 1; 1; 1; n - M\} \\P_{T+2} &= \{1; 1; 1; \ldots; 1; 1; 1; n - M\} \\
P_{T+3} &= \{1; 1; 1; 1; 1; 2; n - M - 1\} \\
P_{T+4} &= \{1; 1; 1; 1; 1; 2; 2; n - M - 1\} \\
P_{T+5} &= \{1; 1; 1; 1; 1; 2; 2; n - M - 2\} \\
& \quad \vdots \\
P_{T+T'-1} &= \{1; 1; 1; 1; 1; 2; 2; n - 2M + 4\} \\
P_{T+T'} &= \{1; 1; 2; 2; 2; 2; n - 2M + 4\} = \{1; 1; 2; 2; 2; 2; 2^{M-1} + l\}
\end{align*}
\]

Hence if we combine two sequences \( \{P_t\}_{t=0}^{T} \) and \( \{P_t\}_{t=T}^{T+T'} \), we can get a sequence \( \{P_t\}_{t=0}^{T+T'} \) from any \((m+1)\)-th stage coalition structure \( P \) to \( P^{**} \). Note that only deviation of a coalition with 2 or more members appears for all \( M \)-th coalition structures in this sequence.

(Step 3) Every coalition structure \( P \) of less than \((2^{m-2}+2)\) coalitions other than the grand coalition structure \( P^N \) is sequentially dominated by \( P^N \).

The proof is similar to that in Theorem 2 except for the cardinality of the key coalition structure \( P^{**} \).

(Step 4) Every coalition structure \( P \) of more than \((2^{m-2}+2)\) coalitions is sequentially dominated by \( P^N \).
The proof is the same as that in Theorem 2.

Steps 1-4 show that every coalition structure other than $\mathcal{P}^N$ is sequentially dominated by $\mathcal{P}^N$. Q.E.D.

We now apply the above theorem when the production function is given by $f(x) = x^\alpha$ ($0 < \alpha < 1$). First of all, by Proposition 3, it is easy to check that for any $\mathcal{P}$

$$x^*_N(\mathcal{P}) = (\alpha + k - 1)(x^*_N(\mathcal{P}))^{\alpha-1}/(kq) = \left(\frac{\alpha - 1 + k}{kq}\right)^{1/(1-\alpha)},$$

$$u_i(\mathcal{P}) = m^*_S(\mathcal{P})/r_1 = [f(x^*_N(\mathcal{P})) - qx^*_i(\mathcal{P})] / (r_1k)$$

$$= [f(x^*_N(\mathcal{P})) - f'(x^*_N(\mathcal{P}))x^*_N(\mathcal{P})] / (r_1k) = (1 - \alpha)(x^*_N(\mathcal{P}))^{\alpha}/(r_1k), \forall i \in S_1.$$

Notice that if $\mathcal{P} = \mathcal{P}^N$, then $k = 1$ and $r_1 = n$, so that

$$x^*_N(\mathcal{P}^N) = \alpha(x^*_N(\mathcal{P}^N))^{\alpha-1}/q = \left(\frac{\alpha}{q}\right)^{1/(1-\alpha)},$$

$$f(x^*_N(\mathcal{P}^N)) - f'(x^*_N(\mathcal{P}^N))x^*_N(\mathcal{P}^N) = (1 - \alpha)(x^*_N(\mathcal{P}^N))^{\alpha}.$$  

This implies

$$B(k) = \{f(x^*_N(\mathcal{P})) - f'(x^*_N(\mathcal{P}))x^*_N(\mathcal{P})\} / [k^2 \{f(x^*_N(\mathcal{P}^N)) - f'(x^*_N(\mathcal{P}^N))x^*_N(\mathcal{P}^N)\}]$$

$$= \frac{1}{k^2} \left(\frac{\alpha - 1 + k}{\alpha k}\right)^{\alpha/(1-\alpha)}.$$

\textbf{Corollary 1.} If $f(x) = x^\alpha$, then for some $\alpha \in (0, 1)$, the grand coalition structure $\mathcal{P}^N$ is sequentially stable for any number of players $n = |N| \geq 4$.

\textbf{Proof.} We will apply Theorem 3. First of all, note that $B(k)$ is an increasing function of $\alpha$, and $\lim_{\alpha \to 0} B(k) = 1/k^2$ for any $k$. Hence for sufficiently small $\alpha > 0$, $B(k)$ is very close to $1/k^2$.

Let $m \geq 2$ be given. Consider any integer $n \in [2^m, 2^{m+1})$. First we will show that $\lim_{\alpha \to 0} B(2^{m-h-1} + 2) = 1/(2^{m-h-1} + 2)^2 < 2^{h-1}/n$ for $h = 1, \ldots, m - 2$. Since $2^{h-1}/n > 2^{h-1}/2^{m+1} = 1/2^{m-h+2}$, it is sufficient to prove that $1/(2^{m-h-1} + 2)^2 < 1/2^{m-h+2}$, that is, $(2^{m-h-1} + 2)^2 \geq 2^{m-h+2}$. If $h \leq m - 4$, then $2^{2(h-m-1)} \geq 2^{m-h+2}$, implying the desired result. Also,

for $h = m - 3$, $(2^{m-h-1} + 2)^2 = (2^2 + 2)^2 > 2^5 = 2^{m-h+2}$,

for $h = m - 2$, $(2^{m-h-1} + 2)^2 = (2 + 2)^2 = 2^4 = 2^{m-h+2}$ and

for $h = m - 1$, $(2^{m-h-1} + 2)^2 = (1 + 2)^2 = 3^2 > 2^3 = 2^{m-h+2}$.

Moreover, $\lim_{\alpha \to 0} B(2) = 1/4 = 2^{m-1}/2^{m+1} < 2^{m-1}/n$. Finally, it is clear that $B(k)$ is decreasing in $k$. Therefore, by Theorem 3, $\mathcal{P}^N$ is sequentially stable for some $\alpha \in (0, 1)$. Q.E.D.
This corollary says that if we apply our stability concept to a common pool resource game, the grand coalition structure can be sequentially stable for any number of players.

Coalition structures other than the grand coalition structure could be sequentially stable. For example, in a 6-person game with \( f(x) = \sqrt{x} \), the coalition structures consisting of \((n - 1)\)-person coalition and one-person coalition, \( \mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\} \) \((\{i\} \in N)\) are also sequentially stable. However, such a coalition structure is quite unfair in the sense that the payoff of the player in one-person coalition is equal to the sum of all other players’ payoffs. We will examine under which condition these undesirable coalition structures are unstable. For \( \mathcal{P} \) with \( |\mathcal{P}| = k \), let \( C(k) \equiv \{f(x^*_N(\mathcal{P})) - f'(x^*_N(\mathcal{P}))x^*_N(\mathcal{P})\}/\{f(x^*_N(\mathcal{P}^{N\setminus\{i\}})) - f'(x^*_N(\mathcal{P}^{N\setminus\{i\}}))x^*_N(\mathcal{P}^{N\setminus\{i\}})\} \).

**Theorem 4.** Let \( n \geq 5 \). If \( C(3) \geq \frac{\sqrt{n-2}}{3} \), then the coalition structures \( \mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\} \), \((\{i\} \in N)\) are not sequentially stable.

**Proof.** We will show that any coalition structure containing three coalitions is not sequentially dominated by \( \mathcal{P}^{N\setminus\{i\}} \) if \( C(3) \geq \frac{\sqrt{n-2}}{3} \). Let \( \mathcal{P} = \{S_1, S_2, S_3\} \), \(|S_1| \leq |S_2| \leq |S_3| \), be a coalition structure containing 3 coalitions.

In any sequence from \( \mathcal{P} \) to \( \mathcal{P}^{N\setminus\{i\}} \), two coalitions must merge into one coalition. Thus it is enough to show that the payoff of each player in one of two coalitions is smaller than the payoff in the coalition \( N \setminus \{i\} \) of \( \mathcal{P}^{N\setminus\{i\}} \). Hence if the largest payoff of a player in the second largest \( S_2 \) among all coalition structures with 3 coalitions is smaller than the payoff of a player in \( N \setminus \{i\} \), we can attain our purpose.

Then we have to compare the payoff \( m_j^*(\mathcal{P}) \) of player \( j \) in a coalition \( S_2 \) of the smallest size with the payoff \( m_j^*(\mathcal{P}^{N\setminus\{i\}}) \).

Remark that such a coalition structure is given by \(|S_1| = 1, |S_2| = |S_3| = \frac{n-1}{2} \) if \( n \) is odd, and \(|S_1| = 1, |S_2| = \frac{n-2}{2}, |S_3| = \frac{n+2}{2} \) if \( n \) is even.

By Proposition 1,

\[ m_j^*(\mathcal{P}) = \left[ f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P}) \right]/(9r_2) \]

for \( j \in S_2 \), and

\[ m_j^*(\mathcal{P}^{N\setminus\{i\}}) = \left[ f(x_N^*(\mathcal{P}^{N\setminus\{i\}})) - f'(x_N^*(\mathcal{P}^{N\setminus\{i\}}))x_N^*(\mathcal{P}^{N\setminus\{i\}}) \right]/(4(n-1)) \]

for \( j \in N\setminus\{i\} \). Note that for \( j \in S_2 \), \( m_j^*(\mathcal{P}) \geq m_j^*(\mathcal{P}^{N\setminus\{i\}}) \) iff \( 4(n-1)/(9r_2) \{f(x_N^*(\mathcal{P})) - f'(x_N^*(\mathcal{P}))x_N^*(\mathcal{P})\}/\{f(x_N^*(\mathcal{P}^{N\setminus\{i\}})) - f'(x_N^*(\mathcal{P}^{N\setminus\{i\}}))x_N^*(\mathcal{P}^{N\setminus\{i\}})\} = 4(n-1)/(9r_2)C(3) \geq 1 \). There are two cases to examine. First, if \( n \) is even, consider a coalition structure \( \mathcal{P} \) with \( r_2 = (n-2)/2 \). In this case, \( 4(n-1)/(9r_2) = \frac{8(n-1)}{9(n-2)} \), so that if \( C(3) \geq \frac{\sqrt{n-2}}{3} \), then \( m_j^*(\mathcal{P}) > m_j^*(\mathcal{P}^{N\setminus\{i\}}) \). Second, if \( n \) is odd, consider a coalition structure \( \mathcal{P} \) with \( r_2 = (n-1)/2 \). In this case, \( 4(n-1)/(9r_2) = \frac{8}{9} \), so that if \( C(3) \geq \frac{\sqrt{2}}{3} \), then \( m_j^*(\mathcal{P}) \geq m_j^*(\mathcal{P}^{N\setminus\{i\}}) \). Q.E.D.

By applying this theorem to the case in which the production function is give by \( f(x) = x^\alpha \) \((0 < \alpha < 1)\), we have the following:
Corollary 2. Let \( n \geq 5 \). If \( f(x) = x^\alpha \) and \( \alpha \geq 0.583804 \), then the coalition structures \( \mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\} \), \( \{\{i\} \in N\} \) are not sequentially stable.

Proof. It is easy to see that

\[
C(3) = \left( \frac{3(\alpha + 1)}{2(\alpha + 2)} \right)^{-\alpha/(1-\alpha)}.
\]

Therefore, \( C(3) > \frac{2}{5} \) iff \( 1/C(3) = \left( \frac{2(\alpha + 1)}{3(\alpha + 2)} \right)^{\alpha/(1-\alpha)} < \frac{5}{8} \). Figure 2 illustrates the function \( 1/C(3) - \frac{5}{8} \). It is not hard to check that if \( 1/C(3) < \frac{5}{8} \) if \( \alpha \geq 0.5083804 \).

The above result shows that for any number of players, the coalition structures \( \mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\} \) cannot be sequentially stable if \( \alpha \) is suitably large.

Remark 1. In our definition of domination, either (i) only two coalitions can merge into one coalition, or (ii) one coalition can break up into two coalitions at each step in a sequence. It is possible to define a slightly different notion of domination such that more than two coalitions are allowed to merge into one coalition at each step in a sequence. Our original concept of sequential stability is a refinement of this alternative notion. For this definition of domination, however, we can prove that the unfair coalition structure \( \mathcal{P}^{N\setminus\{i\}} \) sequentially dominates any other coalition structure for a sufficiently large \( n \).

Proposition 6. Suppose we allow that singleton coalition structure \( \mathcal{P}^I \) can merge into \( \mathcal{P}^{N\setminus\{i\}} \) directly at one step. Given \( \alpha \in (0,1) \), \( \mathcal{P}^{N\setminus\{i\}} \) is sequentially stable for a sufficiently large \( n \).

Proof. \( u_k(\mathcal{P}^{N\setminus\{i\}}) > u_k(\mathcal{P}^I) \) for all \( k \in N\setminus\{i\} \in \mathcal{P}^{N\setminus\{i\}} \) if \( C(n) < \frac{1}{n-1} \), that is,

\[
\left( \frac{2(\alpha - 1 + n)}{(\alpha + 1)n} \right)^{\alpha/(1-\alpha)} \frac{1}{n^2} < \frac{1}{n-1}.
\]

For any \( \alpha \in (0,1) \), this inequality holds if \( n \) is sufficiently large for fixed \( \alpha \). Then under the supposition, \( \mathcal{P}^{N\setminus\{i\}} \) sequentially dominates \( \mathcal{P}^I \). In every coalition structure \( \mathcal{P} \) such that \( |\mathcal{P}| \leq n - 2 \), a member in the largest coalition get more payoff in \( N\setminus\{i\} \) by Proposition 5. Hence one member deviates from the largest coalition in \( \mathcal{P} \). On the other hand, since one-person deviation from the grand coalition is profitable by Lemma 1, \( \mathcal{P}^{N\setminus\{i\}} \) sequentially dominates the grand coalition structure \( \mathcal{P}^N \).

Q.E.D.

Remark 2. Because our sequential domination implies DX-domination, it follows from Corollary 1 that the grand coalition structure can be an EEBA for any number of players if \( |N| \geq 4 \). However, a set of EEBA’s might contain several other coalition structures. In particular, the unfair coalition structure \( \mathcal{P}^{N\setminus\{i\}} = \{\{i\}, N \setminus \{i\}\} \) is an EEBA for a sufficiently large \( n \). In fact, this follows from Proposition 6, because DX-domination is implied by domination under the assumption in Proposition 6 that the singleton coalition structure \( \mathcal{P}^I \) can merge into \( \mathcal{P}^{N\setminus\{i\}} \) directly at one step. It is difficult to eliminate the possibility that the coalition structures \( \mathcal{P}^{N\setminus\{i\}} \) is an EEBA because the singleton player gets the maximal payoff among the payoffs under all coalition structures. (See Diamantoudi and Xue (2002) for a related argument.)
5 Concluding Remarks

We have proposed a sequentially stable coalition structure as a new concept of stability in coalition formation problems. This concept is an extension of EBA coalition structures. We have shown that the grand coalition structure can be sequential stable in common pool resource games.

In this paper, each coalition structure corresponds to one payoff vector. For a more general case in which each coalition structure corresponds to many possible payoff vectors, we have to consider a payoff configuration defined by $(z, \mathcal{P})$, which satisfies $z \in \{ z | z \in \mathcal{F}(\mathcal{P}) \}$. Here $\mathcal{F}(\mathcal{P})$ is a set of feasible payoff vectors under $\mathcal{P}$. In this case, it is not easy to compare the present payoff configuration to the final payoff configuration because of the multiplicity of the final payoff vectors. Then we should take into account sequential domination between two feasible payoff vectors in the same coalition structure. This topic is left for a future research.

We can apply our stability concept to other economic situations like public goods provision games and Cournot oligopoly games. It is generally difficult to check which coalition structures are EBA’s in Cournot oligopoly games (Ray and Vohra (1997)). Examining sequential stability of coalition structures in these economic environments is an open question.

References


Figure 1. A five-person example.
Figure 2. The function $1/C(3)$