Urban Spatial Development: A Real Options Approach

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Abstract

This article investigates urban spatial development in a real options framework where a landowner irreversibly develops property in an uncertain environment. Land that is located far away from the central business district will be developed later, but on a larger scale regardless of whether uncertainty arises or not. However, land development will not continuously move outwards from the central business district under poor demand or supply conditions. It is also found that a landowner will develop property later, but more densely if (i) uncertainty becomes greater; (ii) the returns to undeveloped land are higher; and (iii) the development costs are expected to grow more slowly.

Keywords: Development Density, Real Options, Spatial Development

JEL Classification: G13, R52.
I. Introduction

The traditional literature on urban spatial development (e.g., Turnbull, 1988) has offered insights regarding the pattern of urban development. By assuming that the environment in the real estate market is non-stochastic over time, it predicts that development continuously moves either outward from or inward towards the central business district (henceforth CBD) of a city. Consequently, it precludes the possibility that development may be temporally halted by adverse economic conditions. This article applies the real options model so as to incorporate “uncertainty” in the analysis, which includes the “certainty” case as a polar case.

The traditional literature that offers the “certainty” model also proceeds from the static, to the myopic expectation, and finally to the perfect foresight model. The static long-run equilibrium model, including Alonso (1964), Mills (1967), and Muth (1969), uses the comparative-statics approach, and predicts that residential density declines monotonically with greater distance to the CBD. The prediction portrayed by this kind of model, however, is restricted (Wheaton, 1982). The myopic expectation model such as Anas (1978) assumes that urban development is an incremental process in which development occurs over time in successive zones from the CBD outwards, and that the development of each zone proceeds under myopic foresight. This kind of model produces predictions that are quite different from the
static model. For example, Anas shows that if population grows over time, residential density increases with greater commuting distance to the CBD. One key assumption of the model is that capital investment in developing properties is costlessly reversible. By contrast, the perfect foresight model assumes that capital investment in property development is irreversible due to land use restrictions, and thus a developer must wait for a state of nature that is good enough to develop property. Fujita (1982), McFarlane (1999), and Wheaton (1982) make important contributions to this kind of model, and Turnbull (1988) provides a generalized model in which multiple simultaneous development site locations can arise.

On the other hand, the real options literature (see, e.g., Dixit and Pindyck, 1994) that investigates the timing and density decisions of property development typically abstracts from the spatial factor. For example, Titman (1985) uses a binominal model and demonstrates that uncertainty will delay property development. Clark and Reed (1989) allow a landowner to sequentially choose the timing and density of development, while Williams (1991) allows a landowner to make these two decisions simultaneously. In Capozza and Li (1994), a landowner chooses the timing of development and the capital intensity (defined as capital over land) simultaneously. After solving the optimization problem, they set a pricing equation, which indicates that rents are an increasing function of distance from the edge of the urban area. The
key difference between our article and theirs is that we explicitly place the spatial factor into a landowner’s optimization problem, and thus this factor plays a central role in our analysis. ¹

The remainder of this article is organized as follows. Section II presents the basic model. Section III solves choices regarding the timing and density of development of a landowner for both the certainty and the uncertainty case. Section IV shows the comparative-statics results regarding how the spatial factor, together with various other exogenous forces, affects the timing and the density of development chosen by a landowner. Section V concludes by offering suggestions for future research.

II. The Model

We construct a model that extends both the “certainty” model as in Turnbull (1988) and McFarlane (1999), and the real options model as in Williams (1991). Suppose that at date $t=0$ a risk-neutral landowner has a parcel of vacant land that is normalized at one unit. At any time where $t \geq 0$, the landowner is able to develop vacant land on a scale equal to $Q$, and thus also at a density equal to $Q$. We assume

¹ A minot difference it that their paper assumes that rents follow an arithmatic Brownian motion, while we assume that both rents and the development costs follow joint geometric Brownian motions.
that structures are completely irreversible once put into place. The cost of development is assumed to be equal to (see, e.g., Quigg, 1993; Williams, 1991)

\[ C(Q, x_s(t)) = Q^\eta x_s(t), \quad (1) \]

where the constant cost of scale \( \eta > 1 \) (which indicates that development on a larger scale is more costly), and \( x_s(t) \) is a disturbance term which captures supply shocks such as unexpected changes in weather or labor market conditions.

Following Turnbull (1988, 2005b), we assume that land rent during time \( t \) for the plot of developed property per unit is given by

\[ R(Q, D, x_s(t)) = Q^{b-1}D^{-\alpha}x_s(t), \quad 2 > b > 1, \quad \alpha > 0. \quad (2) \]

In equation (2), the term \( x_s(t) \) denotes the macroeconomic shock from the demand side, and \( D \) is the distance from the CBD, which is the only characteristic that distinguishes parcels of land in a city that is monocentric. Equation (2) indicates that the rent to developed property per unit is increasing concave in structural density (i.e. \( \partial R / \partial Q > 0 \), and \( \partial^2 R / \partial Q^2 < 0 \)), and decreasing in the distance from the CBD (i.e. \( \partial R / \partial D < 0 \)). This indicates that renters are willing to pay less per unit of developed

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\(^2\) McFarlane (1999) argues that investment on land development will be fully irreversible if demolition costs are extremely high. Similarly, Riddiough (1997) suggests that irreversibility is a reasonable assumption with real estate in which the physical asset is long-lived and switching costs to alternative uses are quite high. Turnbull (2005a) argues that the irreversibility assumption may not be realistic, but provides analytically tractable solutions.
property as it is developed more densely or the distance from the CBD is greater.

Both the supply shock, \( x_i(t) \), and the demand shock, \( x_2(t) \), follow joint geometric Brownian motions given by

\[
dx_i(t) = \alpha_i x_i(t) dt + \sigma_i x_i(t) d\Omega_i(t),
\]

(3)

where \( i = 1, 2 \). Each variable \( x_i(t) \) has a constant expected rate of growth \( \alpha_i \) and a constant variance of the growth rate \( \sigma_i^2 \), and each \( d\Omega_i(t) \) is an increment to a standard Wiener process, with \( E[d\Omega_i(t)] = 0 \) and \( E[d\Omega_i(t)^2] = dt \). Furthermore, \( E(d\Omega_i(t)d\Omega_j(t)) = r_{ij}\sigma_i\sigma_j dt \), where \(-1 \leq r_{ij} \leq 1\).

We assume that the risk-less rate of interest \( \rho \) is constant per unit of time and that vacant land per unit earns an agricultural rent, \( \gamma x_i(t) \), which is invariable across space. We further assume that \( \gamma > 0 \) such that a landowner has no incentive to abandon vacant land. In other words, the landowner does not have the option value of abandonment. For tractability, we also abstract from both the time-to-build problem that usually occurs in the real estate industry (see, e.g., Bar-Ilan and Strange, 1996; Capozza and Li, 1994; Grenadier, 2000), and the redevelopment problem addressed in Williams (1997). Consequently, in what follows, the landowner will simultaneously choose the date and the scale of development once and for all.\(^3\)

\(^3\) We also assume that all lots are simultaneously developed and are finished instantly. These assumptions are usually adopted in the real options literature (see, e.g., Capozza and Li, 1994; Childs, et
III. Choices of the Date and the Density of Development

1. The Certainty Case

In what follows, without risk of confusion, we will write \( x(t) = x_1 \) and \( x_1(t) = x_2 \).

The parcel of vacant land belonging to a landowner has the expected value given by

\[
E \left[ \int_{\tau}^{\infty} \gamma x_t(\tau) e^{-\rho(\tau-t)} d\tau + \int_{0}^{\tau} Q^d D^{-\gamma} x_2(\tau) e^{-\rho(\tau-t)} d\tau - Q^d x(T) e^{-\rho(\tau-t)} \right].
\]  

Equation (4) indicates that the expected present value of returns to the unit plot of land at a particular location \( D \) in the urban area is the sum of the expected present value of agricultural rents received until time \( T \), plus the expected present value of land rent beginning at the time of development, less the expected present value of the developing costs. Equivalently, the expected value of vacant land can be rewritten as

\[
\frac{\gamma x_1}{\rho - \alpha} + E \left[ \int_{\tau}^{\infty} (Q^d D^{-\gamma} - \gamma) x_2(\tau) e^{-\rho(\tau-t)} d\tau - Q^d x(T) e^{-\rho(\tau-t)} \right],
\]

where it is required that \( \rho > \alpha \). The first term of equation (4a) is the expected present value of agricultural rents received from \( t \) until infinity, i.e.

\[
E \int_0^{\infty} \gamma x_2(\tau) e^{-\rho(\tau-t)} d\tau = \frac{\gamma x_2}{\rho - \alpha}.
\]

Under the certainty case, \( x_1(T) = x_2 e^{\sigma_1(T-t)} \) and \( x_2(\tau) = x_2 e^{\sigma_2 \tau} \) since \( \sigma_1 = \sigma_2 = 0 \).

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\( \alpha \), 1996; Williams, 1991). We thus neither allow lots to be developed sequentially nor allow the development of real estate as a sequential investment (see, e.g., Bar-Ilan and Strange, 1998).

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The analysis of the “certainty” model can be found in several studies in the real options literature. See, for example, Dixit and Pindyck (1994, pp. 138-139) and Majd and Pindyck (1989).
Substituting this into (4a) yields the objective function of the developer as given by

\[
\max_{T, Q} \left\{ \frac{\gamma x}{(\rho - \alpha_z)} + W(T, Q) \right\},
\]

(4b)

where

\[
W(T, Q) = \frac{(Q^+ D_x - \gamma) x e^{-(\rho - \alpha_z)T}}{(\rho - \alpha_z)} - Q^+ x e^{-(\rho - \alpha_z)T}.
\]

(5)

Intuitively, under the certainty case, a landowner needs to decide either to develop immediately at the current time \( t \) or to delay development. The former occurs if the landowner expects the development costs to grow at a rate that is greater than or equal to the growth rate of the rent for developed property, i.e. \( \alpha_t \geq \alpha_z \).

Begin by assuming that development is made immediately, i.e. \( T = t \), and later we will show that the condition for this to be true is indeed \( \alpha_t \geq \alpha_z \). Substitute \( T = t \) into equation (4a) yields the expected value of vacant land as given by

\[
\frac{Q^+ D_x x}{(\rho - \alpha_z)} - Q^+ x_t.
\]

(4c)

If the value given by equation (4c) is greater than zero, then vacant land will be developed immediately. Assuming this condition holds, we can derive the choice of development scale, denoted by \( Q' \), by differentiating equation (4c) with respect to \( Q \), and then setting it equal to zero. This yields

\[
\frac{bQ^{\alpha_z} D_x x}{(\rho - \alpha_z)} - \eta Q^{\alpha_z} x_t = 0.
\]

(6)

Solving equation (6) yields

\[
Q' = \left( \frac{b x_t}{\eta(\rho - \alpha_z) D_x x} \right)^{1/(\eta \alpha_z)}.
\]

(7)
Evaluating equation (4c) at $Q = Q'$, and using equation (6) yields the optimized expected value of vacant land as given by

$$\left(\frac{\eta}{b} - 1\right)Q'^n x_i.$$  \hspace{1cm} (4d)

The value given by equation (4d) will be negative if $b > \eta$, i.e. when the scale of development is increased by one percent, total rent will be increased at a rate which is larger than that of the development costs. If this holds, then development will never occur, regardless of the order of $\alpha_i$ and $\alpha_z$. If instead, $\alpha_i > \alpha_z$ and $\eta > b$, then land will be developed immediately.

When $\alpha_i > \alpha_z$, the optimal development strategy for a landowner is the “now-or-never” strategy. By contrast, it is better for the landowner to delay development under the scenario where $\alpha_i < \alpha_z$ and $\eta > b$. In this case, we denote $T'$ and $Q'$ as the interior solutions for the timing of development, $T$, and the scale of development, $Q$, respectively. They can be derived by partially differentiating equation (5) with respect to $T$ and $Q$, respectively, and then setting the results equal to zero, i.e.

$$\frac{\partial W(T', Q')}{\partial T} = \gamma x_i e^{(\rho - \alpha_i)T'} + (\rho - \alpha_i)Q'^n x_i e^{(\rho - \alpha_i)T'} - Q'^b D^n x_i e^{(\rho - \alpha_i)T'} = 0,$$  \hspace{1cm} (8)

$$\frac{\partial W(T', Q')}{\partial Q} = bQ'^b D^n x_i e^{(\rho - \alpha_i)T'} \left(\rho - \alpha_z\right) - \eta Q'^n x_i e^{(\rho - \alpha_i)T'} = 0.$$  \hspace{1cm} (9)

Equation (8) shows the condition for the choice of timing of development. The
landowner must wait until an appropriate date to develop vacant land. By postponing the development process, the developer gains the agricultural rents (the first term on the right-hand side), and saves the annualized development cost that would be put into place when the land is developed (the second term on the right-hand side). At the same time, any postponement of the development time beyond a certain period means that a developer forgoes what the land would have earned as developed during that period (the last term on the right-hand side). The best time to develop is when the marginal benefit from and marginal cost of waiting are equal. Equation (9) simply says that the marginal benefit from developing vacant land must be equal to the marginal cost of developing it. Both terms are the first and second terms on the right-hand side of equation (9), respectively.

The second-order conditions are given by

\[ \frac{\partial^2 W(T^*, Q^*)}{\partial T^2} = - (\alpha_z - \alpha_r)(\rho - \alpha_r)Q^{\alpha_r}e^{-(\rho-\alpha_r)T^*} < 0, \]

\[ \frac{\partial^2 W(T^*, Q^*)}{\partial Q^2} = - (\eta - \theta)\eta Q^{\alpha_r-1}x^*e^{-\theta x^*} < 0. \]

The second-order condition with respect to \( T \) implies that for there to be a local maximum, i.e. for obtaining a finite and non-zero value of \( T \), rents for developed properties must grow more rapidly than the development costs (\( \alpha > \alpha_r \)). The second-order condition with respect to \( Q \) must be negative, which requires that incremental rents cut the incremental costs of development from above, or equivalently,
\[ \frac{\partial^2 W(\cdot)}{\partial T^2} \cdot \frac{\partial^2 W(\cdot)}{\partial Q^2} - \left( \frac{\partial^2 W(\cdot)}{\partial T\partial Q} \right)^2, \]
is positive. We find that this holds if \( \alpha_i > \alpha_i \).

Solving equations (8) and (9) simultaneously yields

\[ T' = t + \frac{1}{(\alpha_i - \alpha_i)} [\ln \eta + \ln(\rho - \alpha_i) + \frac{an}{b} \ln D + (\frac{\eta}{b} - 1) \ln g_i - \ln x_i + \ln x_j], \quad (12) \]

\[ Q' = g_i D^{\xi}, \quad (13) \]

where \( g_i = \gamma [1 - \frac{b(\rho - \alpha_i)}{\eta(\rho - \alpha_i)}]^{\frac{1}{\rho}}. \quad (14) \]

If \( \alpha_i > \alpha_i \) and \( \eta > b \), then the landowner will wait until \( T' \) in equation (12) is reached. In other words, even under the certainty case, a landowner has an option value of waiting since it is better for him to delay development. By contrast, as mentioned before, if \( \alpha_i \geq \alpha_i \), then it is better for the landowner to adopt the “now-or-never” strategy.

2. The Case of Uncertainty

We will now return to the general case in which the overall volatility \( \sigma \neq 0 \).\(^6\)

The problem is to determine the point at which it is optimal to invest so as to maximize the expected value of the vacant land. Since this value evolves stochastically, we will

\(^5\) For \( g_i \) to be non-negative, it is also required that \( \eta(\rho - \alpha_i) > b(\rho - \alpha_i) \).

\(^6\) This condition allows either \( \sigma_i \) or \( \sigma_j \) to be zero, but does not allow both \( \sigma_i \) and \( \sigma_j \) to be zero simultaneously.
not be able to determine a time \( T \) as we did in the certainty case (see, e.g. Dixit and Pindyck, 1994, pp. 139-140). Instead, our investment rule will take the form of a critical value of \( x_1^* \) and \( x_2^* \) such that it is optimal to develop properties on a scale equal to \( Q^* \) once \( x_2 > x_2^* \) and \( x_2 < x_1^* \). The problem of the developer is thus equivalent to

\[
\max_{x_1, x_2, Q} \left\{ \frac{\gamma x_1}{(p - \alpha_1)} + V_e(x_1, x_2, Q) \right\},
\]

where

\[
V_e(x_1, x_2, Q) = E \int_0^T (Q^D_x^e - \gamma)x_2(\tau)e^{-\rho(\tau - t)}d\tau - Q^*x_2(T)e^{-\rho(T - t)},
\]

and \( \gamma x_1 / (p - \alpha_1) \) is the value of vacant land if undeveloped forever.

The Bellman-Hamilton-Jacobi equation for the problem can be written as

\[
pV_e(x_1, x_2, Q) = \max_{x_1, x_2, Q} \left\{ (Q^D_x^e - \gamma)x_2 - (p - \alpha_1)Q^*x_2 + E_t \frac{dV_e(x_1, x_2, Q)}{dt} \right\}.
\]

We can treat \( V_e(\cdot) \) as the value of an asset. Equation (17) then says that in order to prevent arbitrage profits from arising, the total return on the asset, \( pV_e(\cdot) \), must be equal to the sum of two components: the income flow \( (Q^D_x^e - \gamma)x_2 - (p - \alpha_1)Q^*x_2 \), and the expected rate of capital gain given by

\[
E_t \frac{dV_e(\cdot)}{dt} = \frac{1}{2} \sigma^2 x_1 \frac{\partial^2 V_e(\cdot)}{\partial x_1^2} + r_t \sigma x_1 x_2 \frac{\partial^2 V_e(\cdot)}{\partial x_1 \partial x_2} + \frac{1}{2} \sigma^2 x_2 \frac{\partial^2 V_e(\cdot)}{\partial x_2^2} + \alpha x_1 \frac{\partial V_e(\cdot)}{\partial x_1} + \alpha x_2 \frac{\partial V_e(\cdot)}{\partial x_2}.
\]

Substituting equation (18) into equation (17) yields
\[
\rho V_x = \max_{\lambda, \sigma, \rho} \left\{ \frac{1}{2} \sigma^2 x_t^2 \frac{\partial^2 V_x}{\partial x_t^2} + r_t \sigma x_t \frac{\partial^2 V_x}{\partial x_t \partial x_{t-1}} + \frac{1}{2} \rho_t^2 \frac{\partial^2 V_x}{\partial x_t^2} \right\}
\]

(19)

Rearranging the terms in equation (19) yields the following differential equation:

\[
\frac{1}{2} \sigma^2 x_t^2 \frac{\partial^2 V_x}{\partial x_t^2} + r_t \sigma x_t \frac{\partial^2 V_x}{\partial x_t \partial x_{t-1}} + \frac{1}{2} \rho_t^2 \frac{\partial^2 V_x}{\partial x_t^2} + \alpha x_t \frac{\partial V_x}{\partial x_t} + \alpha x_t(t) \frac{\partial V_x}{\partial x_{t-1}} + (Q'D^\gamma - \gamma)x_t - (\rho - \alpha)Q^\gamma x_t = 0.
\]

(19')

One particular solution from the non-homogeneous part of equation (19') is:

\[
V_x(x_t, x_{t-1}, Q) = \frac{(Q'D^\gamma - \gamma)x_t}{(\rho - \alpha)} - Q^\gamma x_t.
\]

(20)

The term \( V_x(.) \) is the net value from developing vacant land immediately since, as the landowner develops vacant land immediately, he receives the expected present value of the land rents beginning from time \( t \), i.e.

\[
E\int Q'D^\gamma x_t(\tau)e^{-\rho t}d\tau = \frac{Q'D^\gamma x_t}{(\rho - \alpha)}.
\]

but scarifies the expected present value of agricultural rents received until infinity, i.e. \( \gamma x_t/(\rho - \alpha) \) given by equation (5), as well as the developing costs, \( Q^\gamma x_t \). Based on Bertola and Caballero (1994, Appendix), the term \( x_t^\beta x_{t-1}^{1-\beta} \) solves the homogeneous part of equation (19'). Substituting this into equation (19') yields the quadratic equation given as

\[
\varphi(\beta) = -\frac{\sigma^2}{2} \beta(\beta - 1) - \beta(\alpha - \alpha_t) + (\rho - \alpha_t) = 0,
\]

(21)

where \( \sigma^2 = \sigma^2_t - 2r_t \sigma_t \sigma_x + \sigma_x^2 \), which is defined as the “overall volatility.” The
general solution to equation (19'), which is composed of solutions from both the homogeneous and non-homogeneous parts of equation (19'), is thus given by

\[ V_d(x_1, x_2, Q) = A_1 x_1^{r_1} x_2^{r_2} + A_2 x_2^{r_2} x_1^{r_1} + V'(x_1, x_2, Q), \]  

(22)

where \( A_1 \) and \( A_2 \) are constants to be determined, and \( \beta_1 \) and \( \beta_2 \) are respectively the larger and smaller roots of the quadratic equation given by equation (21).

A developer simultaneously chooses the timing and density of development. The timing decision is characterized by \( x' \) and \( x'_2 \), both of which are the respective values of \( x_1 \) and \( x_2 \) that trigger property development. The density decision is characterized by \( Q' \), which is the value of \( Q \) chosen by the developer. These three critical values, together with \( A_1 \) and \( A_2 \) in equation (22), are solved from the equations given by:

\[ \lim_{x_i \to 0} V_d(x_1, x_2, Q) = 0, \]  

(23)

\[ V_d(x'_1, x'_2, Q') = 0, \]  

(24)

\[ \frac{\partial V_d(x'_1, x'_2, Q')}{\partial x_i} = 0, \]  

(25)

\[ \frac{\partial V_d(x'_1, x'_2, Q')}{\partial x_2} = 0, \]  

(26)

\[ \frac{\partial V_d(x'_1, x'_2, Q')}{\partial Q} = 0. \]  

(27)

Equation (23) is the boundary condition, which states that the option value of developing vacant land is worthless as the demand-shift factor approaches its
minimum permissible value of zero. Equation (24) is the value-matching condition which states that, at the optimal timing of development, a landowner should be indifferent as to whether vacant land is developed or not. Equations (25) and (26) are the smooth-pasting conditions, which require that the landowner not obtain any arbitrage profits from deviating from the optimal timing of development. Equation (27) indicates that the chosen density should maximize the net value of developing land immediately, which in turn requires that the marginal benefit from developing vacant land be equal to the marginal cost of developing it.

Equations (23)-(27) are satisfied by the value function $V_j(\cdot)$ that is linearly homogeneous in $x_j$ and $x_z$, and thus we can write $v_j(y, Q) = V_j(x_j, x_z, Q) / x_j$, and $v_j(y, Q) = V_j(x_j, x_z, Q) / x_j$ (see, e.g., Williams, 1991), where we define $y = x_j / x_i$. Note that a higher value of $y$ indicates that the state of nature is better because it comes from a larger value of $x_z$ and/or a smaller value of $x_i$, i.e. when demand for developed property is increased and/or the cost of developing vacant land is reduced.

Equations (23)-(27) can thus be rewritten as

$$
\lim_{y \to 0} v_j(y, Q) = 0,
$$

(28)

$$
v_j(y', Q') = 0,
$$

(29)

$$
\frac{\partial v_j(y', Q')}{\partial y} = 0,
$$

(30)
\[
\frac{\partial v(y', Q')}{\partial Q} = 0,
\]

where \( y' = x'_r / x'_i \) represents the choice of development timing. Solving equation (28) yields \( A_i = 0 \). Solving equation (30) yields

\[
A_i = \frac{-\left( Q^r D^r - \gamma \right) y'^{i-\eta}}{\beta_i (\rho - \alpha_i)}.
\]

Substituting the values of \( A_i \) and \( A_j \) into equation (29) and referring to the result as

\[
M(y', Q') \text{ yields}
\]

\[
M(y', Q') = -\left(1 - \frac{1}{\beta_i}\right)(Q^r D^r - \gamma) \frac{y'}{(\rho - \alpha_j)} + Q'^{\eta} = 0.
\]

Equation (32) states that a developer should postpone development until the conversion value of a parcel of agricultural land changed to residential usage is just equal to the value of the construction costs. In addition, denoting the partial derivative results of equation (31) as \( S(y', Q') \) yields

\[
S(y', Q') = \frac{h Q^r D^r y'}{(\rho - \alpha_j)} - \eta Q'^{\eta+1} = 0.
\]

Equation (33) states that the marginal benefit from developing vacant land must be equal to the marginal cost of developing it. Solving equations (32) and (33) simultaneously yields

\[
y' = \frac{\eta(\rho - \alpha_j)}{b} g_i^{\frac{-1}{\beta_i}} D^r \frac{\gamma}{r},
\]

\[
(34)
\]

\footnote{Note that once a landowner develops property, then he kills the option value of developing vacant land. This is shown by the negative value of \( A_i \). Consequently, the term \( V_s(x_i, x_j, Q) \) shown by equation (22) is equal to the net value from developing immediately, \( V_s(x_i, x_j, Q) \), net of the option value from developing vacant land immediately, \( A_i x^r_i x^{i-\eta} + A_j x^r_j x^{i-\eta} \), where \( A_i < 0 \) and \( A_j = 0 \).}
\[ \frac{Q'}{g_i} = \frac{\gamma}{\beta} D^\gamma, \]  

(35)

where

\[ g_i = \gamma \left[ 1 - \frac{b}{\eta} \left( \frac{1}{\beta} \right)^\gamma \right]^\gamma. \]  

(36)

We can compare our results with those of the “certainty” model in the literature such as in Turnbull (1988). To compare our results with the “certainty” case, we can find an expression for \( y \) that triggers land development immediately, i.e. that makes \( T' \) derived in equation (12) equal to the current time \( t \). Consequently, this critical level of \( y \), still defined as \( y' \), is the \( y \) that makes the terms inside the brackets on the right-hand side of equation (10) equal to zero. This yields

\[ y' = \frac{\eta(p - \alpha_z)}{b} g^{\gamma} D^\gamma. \]  

(3)

The “certainty” model typically assumes that a landowner has the option value to delay development. This is equivalent to our case where the following assumptions hold: \( \alpha_z > \alpha_i \), \( b > \eta \), and the overall volatility \( \sigma \) approaches zero. Given this case, the larger root of equation (21), \( \beta_i \), will approach \((p - \alpha_i)/(\alpha_z - \alpha_i) > 1 \) since \( p > \alpha_z \).

Substituting this into equation (36) yields \( g_i = \gamma [1 - \frac{b(p - \alpha_z)}{\eta(p - \alpha_i)}]^\gamma \), which is exactly the same as \( g_i \) defined in equation (14) such that the solution of \( y' \) and \( Q' \) collapses to its counterparts in the certainty case. In other words, the analysis of the “certainty”
model in Section III.1 is just a special case of the real options model in this section.

We have obtained analytically tractable solutions for both the optimal date, \( y' \), and the optimal density, \( Q' \). However, to gain more insights regarding how the underlying exogenous forces affect \( y' \) and \( Q' \), we will focus on both the optimal condition for deriving \( y' \), and that for deriving \( Q' \). Equation (32) implicitly defines the positive dependence of \( y' \) on \( Q' \), and equation (33) implicitly defines the positive dependence of \( Q' \) on \( y' \). Totally differentiating equation (32) with respect to \( Q' \), and using equations (34)-(36) yields

\[
\frac{\partial y'}{\partial Q'} = \frac{\Delta_{i_2}}{-\Delta_{i_1}} > 0, \tag{38}
\]

where

\[
\Delta_{i_1} = \frac{\partial M(y', Q')}{\partial y} = -(1 - \frac{1}{\beta_i}) \frac{(Q'^+D^{-\gamma})}{(\rho - \alpha_x)} < 0, \tag{39}
\]

\[
\Delta_{i_2} = \frac{\partial M(y', Q')}{\partial Q} = \frac{bQ'^+D^{-\gamma}y'}{\beta_i(\rho - \alpha_x)} > 0. \tag{40}
\]

Totally differentiating equation (33) with respect to \( y' \), and using equations (34)-(36) yields

\[
\frac{\partial Q'}{\partial y'} = \frac{\Delta_{i_2}}{-\Delta_{i_2}} > 0, \tag{41}
\]

where

\[
\Delta_{i_2} = \frac{\partial S(y', Q')}{\partial Q} = -\eta(\eta - 1)Q'^{\gamma - 2} < 0, \tag{42}
\]
The Jacobian condition requires that

\[ \Delta_{ii} \Delta_{ii} - \Delta_{ii} \Delta_{ii} > 0. \]  

We depict the impact of \( Q \) on \( y \) in equation (38), and that of \( y \) on \( Q \) in equation (41) by line \( MM \) and line \( SS \) in Figure 1, respectively. The positive slope of line \( MM \) indicates that as the scale of development widens, a landowner will not develop until the state of nature becomes better. The positive slope of line \( SS \) indicates that a landowner will develop vacant land on a larger scale as the state of nature becomes better. In addition, equation (44) requires that the slope of \( SS \) be steeper than that of \( MM \), and we find that this requirement is satisfied.

IV. Comparative-Statics Results

Proposition 1 stated below indicates how a change in the distance from the CBD affects a landowner’s choice of the date on which and density at which to develop.

**Proposition 1:** A landowner will develop later but more densely on land that is located far away from the CBD (i.e. \( D \) is larger).

Proof: See Appendix A.
We explain the intuition behind Proposition 1 by using Figure 1. Suppose that
the initial equilibrium is at point A, where the landowner chooses a density equal to
$Q_s$, and a date of development equal to $y_o$. The effect of a rise in the distance from
the CBD on the timing of development $y'$ and the density of development $Q'$
combines the two effects stated below. First, given the density of development, the
vacant land that is located far away from the CBD will have lower rents after
development. Consequently, the landowner will delay developing, because waiting
will then be more valuable, as indicated by equation (A4) (this is shown by the line
$MM$ that shifts upward to line $M'M'$). This, in turn, induces the landowner to
develop property more densely because the landowner will be bold if the state at which
he develops land is good enough. The equilibrium point thus moves from A along
line $SS'$ to point B, where the landowner chooses a density equal to $Q_i$ ($>Q_s$), and a
date of development equal to $y_i$ ($>y_o$). Second, as shown by equation (A6), given
the timing of development, a landowner perceives that the marginal benefit from
developing will decrease as land is located far away from the CBD, such that the
landowner develops less densely (this is shown by the line $SS$ that shifts leftward to
line $S'S'$). This, in turn, induces the landowner to develop earlier because waiting
will then be less valuable as land rents, $Q$ multiplied by $R(\cdot)$ given by equation (2),
will be lower when he develops less densely. The equilibrium point thus moves from
B along line $M'M'$ to point C. As compared to the initial equilibrium point A, at point C the landowner will develop at a larger density ($Q > Q_c$) and develop later ($y > y_c$), as shown by equations (A2) and (A1), respectively.

We may compare our results with those of the “certainty” model such as in Turnbull (1988) and McFarlane (1999). Turnbull constructs a generalized model in which no explicit functional forms are imposed on the demand and supply conditions of the real estate market. Turnbull then finds that whether or not the demanded density is increasing over time is important in determining how the distance from the CBD affects the timing and density of development. In particular, if it is increasing, then the statement in our Proposition 1 will hold.\(^8\) McFarlane (1999) constructs a model in which a Cobb-Douglas production technology that links housing service with land and capital inputs is imposed, while no functional forms are imposed on the demand condition of the real estate market. McFarlane then finds that the greater the distance from the CBD, the later the parcel of land will be developed. McFarlane also finds that the impact of the distance from the CBD on the timing and density of development depends on a growth factor $\phi$: density will increase when $\phi < 1$. We

\(^8\) In Turnbull’s article, the demanded density that is increasing over time is equivalent to $\pi c > 0$. If this is the case, the two comparative-statics in equation (9) in his article will also be positive, i.e. the statement of our Proposition 1 holds.
find that the restriction that $\phi < 1$ implies that the demanded density is increasing over
time (but not vice versa) (see Appendix B). Consequently, our results stated in
Proposition 1 come from the special functional forms we adopt, which implies that
land will be developed more densely at a better state of nature, as indicated by equation
(41). Our result is thus not general enough in the viewpoint of Turnbull (2005a).
However, our specific assumption helps us derive unambiguous comparative-statics
results that can be empirically tested.

The next Proposition states how the other exogenous variables affect a
landowner’s choice of the timing and density of development.

**Proposition 2:** A landowner will develop property later, but more densely, if (i) the
overall volatility is increased ($\sigma$ is increased); (ii) the landowner expects the
development costs to grow more slowly ($\alpha$, is decreased); and (iii) the landowner
expects the returns to undeveloped land per unit to be higher ($\gamma$ is increased).

Proof: See Appendix C.
Suppose that we still use point A in Figure 1 as the initial equilibrium point. Given the three premises stated in Proposition 2, the line $SS'$ will remain unchanged (as shown by equation (C3)), but the line $MM'$ will shift upward to $M'M'$ (as shown by equations (C4)-(C6)), and thus the new equilibrium is at point B, where the date on which to develop is increased from $y_a$ to $y_i$ and the development density is raised from $Q_a$ to $Q_i$. Intuitively, under the first and second scenarios, the landowner had better delay development because waiting will then become more valuable (or equivalently, the first two terms of $V_i(\cdot)$ defined in equation (22) will be higher). As for the third scenario, a landowner will lose more agricultural rents once he develops vacant land. In other words, he will gain less from developing immediately (or equivalently, $V_i(\cdot)$ defined in equation (20) will be lower). Consequently, under the three scenarios, a landowner will develop vacant land later, but with a higher density as the state of nature at which the landowner develops will become better. Note that only the first scenario is peculiar to the “uncertainty” model, while the other two scenarios will also hold even if we employ the “certainty” model.

We have already dealt with the optimal development strategy $(y', Q')$ at a given location $D$. We can follow the “certainty” model (e.g., Fujita, 1982) to ask the following question: Given a state of nature denoted by $y(t)$, where will the location $D'(y(t))$ at which the optimal development takes place in that state be, and what will
be the optimal lot size $Q'(y(t))$ to be constructed there in that state? We can rewrite equations (34) and (35) as follows:

$$D'(y(t)) = \left( \frac{by(t)g_{\eta_1}}{\eta(\rho - \alpha_2)} \right)^{\frac{1}{\eta_1}}, \quad (45)$$

$$Q'(y(t)) = \left( \frac{by(t)g_{\eta_2}}{\eta(\rho - \alpha_2)} \right)^{\frac{1}{\eta_2}}. \quad (46)$$

If the state of nature $y(t)$ and the distance from the CBD is such that $y(t)$ is greater than the $y'$ given by equation (34), then some landowners will develop land further away from the CBD, increasing $D$ until $y'$ rises to $y(t)$. Equivalently, if the distance from the CBD, $D$, is less than the $D'(y(t))$ given by equation (45), then some landowners will develop further away from the CBD so as to obtain $D = D'(y(t))$; otherwise no action should be taken.\(^9\)

We can compare the development pattern of the “uncertainty” model with that of the “certainty” model. For the “certainty” case, land will be developed now or later if incremental rents cut the incremental costs of development from above, i.e. $\eta > b$ as mentioned before. Let us assume that this requirement holds. If the construction costs grow at a rate that is not less than that of land rents after development, i.e. $\alpha_1 \geq \alpha_2$, then all parcels of vacant land will be developed immediately. Equation (14)

\(^9\) The above argument resembles that in Pindyck (1988) when he explains the timing decision for undertaking a continuous investment project.
then indicates that the greater the distance of a parcel of land from the CBD, the lower
the density at which the parcel of land will be developed. However, if \( \alpha_z > \alpha_s \), then
land will be developed continuously outward from the CBD and the density of
development will increase monotonically the greater the distance to the CBD. This is
because, as indicated by equation (37), the further a parcel of land is located from the
CBD, the later that parcel of land will be developed. In addition, as indicated by
equation (11), the further a parcel of land is from the CBD, the higher will be the
density at which the parcel of land will be developed. By contrast, in our framework
where uncertainty exists in the real estate market, land will not be developed until the
state of nature \( y(t) \) reaches \( y^* \) as defined in equation (34). Consequently, land
development can be temporarily halted if the state of nature becomes worse due to
poor demand and/or supply conditions, i.e. due to a lower value of \( x_z(t) \) and/or a
higher value of \( x_s(t) \) that results in a lower value of \( y(t) \).

We establish a set of parameter values to demonstrate the theoretical results stated
above. Following Dixit (1989) and Williams (1991), the benchmark parameter values
are chosen as follows: the cost elasticity of scale \( \eta = 2 \), the price elasticity of demand
\( 1/(b-1) = 2 \) \( (b = 1.5) \), and the magnitude of the negative effect of the distance from the
CBD on rent \( \alpha_s = 0.2 \); the expected growth rates of the unit construction cost and the
demand shock are respectively given by \( \alpha_z = 0 \) and \( \alpha_s = 1\% \) per year; the instantaneous
volatilities of these two growth rates are given by \( \sigma_1 = \sigma_2 = 20\% \) per year; the correlation coefficient between these two growth rates \( r_{y_{y}} = 0 \), the risk-less rate \( \rho = 5\% \) per year, and the return on undeveloped land per unit \( \gamma = 0.2 \). Given these parameter values, the variable \( y(t) \), which is equal to \( x_y(t)/x_y(t) \), evolves as a geometric Brownian motion with a drift rate equal to \( \alpha_2 - \alpha_1 = 1\% \) per year, and an instantaneous volatility \( \sigma \) equal to \( (\sigma_1^2 + \sigma_2^2 - 2r_{y_{y}} \sigma_1 \sigma_2)^{1/2} = 20\% \) per year. Figure 2 shows a simulation series for \( y(t) \) with an initial value of \( y(0) = 0.1 \).

Let us start from \( t = 0 \). Define \( D_x \) as the \( D \) that makes \( y(0) = y'(D_x) \). As shown by Figures 3 and 4, the parcels of vacant land whose distance from the CBD is less than \( D_0 = 1.39 \) will be developed immediately on a scale not larger than 3.09. Figure 4 then shows the density of development corresponding to different locations in relation to the CBD. For ease of exposition, we classify the process of spatial development of an open city into four regions in date order; in each region development will first start because the economic conditions are better than before, but will then stop due to adverse economic conditions. These regions are: (i) \( 0 < t \leq 0.71 \) (year); (ii) \( 0.71 \) (year) \( < t \leq 1.79 \) (year); (iii) \( 1.79 \) (year) \( < t \leq 9.74 \) (year); and (iv) \( 9.74 \) (year) \( < t \leq 10 \) (year). Let us take region (i) as an example. When the state of nature is better than before (i.e. \( y(t) \) goes upward, as shown by Figure 2), then a parcel of land whose distance from the CBD is between 1.39 and 1.52 will be developed, as
shown by Figure 3, while the corresponding density of development will be between 3.09 and 3.12, as shown by Figure 4. However, when the state of nature is not better than before (i.e. $y(t)$ goes downward, as shown by Figure 2), no more parcels of vacant land will be developed. Such periods (not fully shown in the figures) extend from year 0.04 until year 0.06, year 0.11 until year 0.23, year 0.28 until year 0.32, or from year 0.43 until year 0.70, as shown by the horizontal paths in both Figures 3 and 4. Regions (ii) to (iv) also have similar patterns of development to those in region (i). For example, in region (iii), from year 2.34 until year 9.73, no more parcels of vacant land will be developed because the state of nature is never better than that in year 2.34 (where $y=0.1103$, not shown in the figures). Consequently, unlike the “certainty” model, our model can explain why development may not proceed all the time.

VI. Conclusion

This article investigates urban spatial development in a real options framework where a landowner irreversibly develops property in an uncertain environment. Land that is located far away from the CBD will be developed later, but on a larger scale regardless of whether uncertainty arises or not. However, the development of land
will not continuously move outwards from the CBD under poor demand or supply conditions. It is also found that a landowner will develop property later, but more densely if (i) uncertainty becomes greater; (ii) the returns to undeveloped land are higher; and (iii) the development costs are expected to grow more slowly.

Our article could at least be extended in the following way. First, our article assumes that the cash flow from the undeveloped properties is always positive, thus precluding the abandonment option for these properties. As Williams (1991) suggests, these properties may yield negative cash flows if we make an allowance for the costs of maintaining them. Second, our article assumes that rents for developed properties are affected by the spatial factor and the density of development, while abstracting from the other important factors such as population density and residential income level (see, e.g., Anas, 1978). We may take these extensions into account, and then investigate whether our conclusion regarding the impacts of the spatial factor on a landowner’s choices of the timing and density of development will still hold or not.
Appendix A:

Totally differentiating both $y'$ in equation (34) and $Q'$ in equation (35) with respect to $D$ yields

$$\frac{dy'}{dD} = \frac{\partial y'}{\partial D} + \frac{\partial Q'}{\partial D} \frac{\partial Q'}{\partial y'} > \eta = \frac{a y'}{b D} > 0, \quad (A1)$$

$$\frac{dQ'}{dD} = \frac{\partial Q'}{\partial D} + \frac{\partial Q'}{\partial y'} \frac{\partial y'}{\partial D} = \frac{a Q'}{b D} > 0, \quad (A2)$$

where $\frac{\partial y'}{\partial D} = \frac{\Delta_{11}}{-\Delta_{11}} > 0$, \quad (A3)

$$\Delta_{11} = \frac{\partial M(y', Q')}{\partial D} = (1 - 1) \frac{a Q' D x y'}{(\rho - \alpha_z) > 0}, \quad (A4)$$

where $M(\cdot)$ is defined in equation (32).

$$\frac{\partial Q'}{\partial D} = \frac{\Delta_{31}}{-\Delta_{31}} < 0, \quad (A5)$$

$$\Delta_{31} = \frac{\partial S(y', Q')}{\partial D} = -ab Q' x y' \frac{D x y'}{(\rho - \alpha_z) < 0}, \quad (A6)$$

where $S(\cdot)$ is defined in equation (33). This completes the proof.

Appendix B:

In equation (6) of McFarlane’s (1999) article, the hurdle rent is expressed as

( using the same notations as in his article)

$$R(z, T) = \frac{1}{1 - \gamma} [\gamma G(T) + \frac{A}{q(k)}], \quad (B1)$$
where the present value of future growth in rents \( G(T) = \int_0^T R(z,s)e^{-r(T-s)}ds \). (B2)

Differentiating equation (B1) with respect to \( T \) yields

\[
R_z(z,T) = \frac{\gamma}{1-\gamma} G'(T), \quad 0 < \gamma < 1, \tag{B3}
\]

where \( G'(T) = -R_z(z,T) + rG(T) \geq 0 \) if \( \phi(T) \leq 1 \). (B4)

since \( \phi(T) = \frac{R_z(z,T)}{rG(T)} \). This completes the proof.

Appendix C:

Let \( h \) denote \( \sigma \), \( \gamma \), and \( \alpha \), respectively. Totally differentiating both \( y' \) in equation (34) and \( Q' \) in equation (35) with respect to \( h \) yields

\[
\frac{dy'}{dh} = \frac{\partial y'}{\partial h} + \frac{\partial y'}{\partial \gamma} \frac{\partial \gamma}{\partial h} \frac{\partial \gamma}{\partial h} > (0) \quad \text{when} \; h = \sigma \; \text{or} \; \gamma(\alpha), \tag{C1}
\]

\[
\frac{dQ'}{dh} = \frac{\partial Q'}{\partial h} + \frac{\partial Q'}{\partial \gamma} \frac{\partial \gamma}{\partial h} \frac{\partial \gamma}{\partial h} > (0) \quad \text{when} \; h = \sigma \; \text{or} \; \gamma(\alpha), \tag{C2}
\]

since \( \frac{\partial y'}{\partial \gamma} > 0, \frac{\partial Q'}{\partial \gamma} > 0, \frac{\partial Q'}{\partial \gamma} = \frac{\partial S()}{\partial h} = 0, \tag{C3}
\]

and \( \frac{\partial y'}{\partial h} = \frac{\partial M()}{\partial h} \),

where

\[
\frac{\partial M()}{\partial \sigma} = -1 \frac{\partial \beta_i}{\partial \sigma} \frac{y'}{\beta_i} (r - \alpha_i) [Q^+ D^+ - \gamma] > 0, \tag{C4}
\]

\[
\frac{\partial M()}{\partial \alpha_i} = -1 \frac{\partial \beta_i}{\partial \alpha_i} \frac{y'}{\beta_i} (r - \alpha_i) [Q^+ D^+ - \gamma] < 0, \tag{C5}
\]

\[
\frac{\partial M()}{\partial \gamma} = (1 - \frac{1}{\beta_i}) \frac{y'}{\beta_i} (r - \alpha_i) > 0, \tag{C6}
\]

since \( Q^+ D^+ > \gamma \), \( \partial \beta_i / \partial \sigma < 0 \), and \( \partial \beta_i / \partial \alpha_i > 0 \).
REFERENCES


Figure 1: A rise in $D$. 
Figure 2: A Simulation Series of the states of nature $y(t)$ with $y(0) = 0.1$.

Figure 3: The Optimal Distance $D^*(y(t))$ from CBD to be cultivated.

Figure 4: The Optimal Density of Development $Q^*(y(t))$. 

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